

# Non-Kähler heterotic rotations

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## Abstract

We study a supersymmetry-preserving solution-generating method in heterotic supergravity. In particular, we use this method to construct one-parameter non-Kähler deformations of Calabi-Yau manifolds with a  $U(1)$  isometry, in which the complex structure remains invariant. We explain how to obtain corresponding solutions to heterotic string theory, up to first order in  $\alpha'$ , by a modified form of the standard embedding. In the course of the paper we also show that Abelian heterotic supergravity embeds into type II supergravity, and note that the solution-generating method in this context is related to a dipole-type deformation when there is a field theory dual.

## 1 Introduction

Supersymmetric solutions of the common sector of supergravity theories in ten dimensions have attracted the attention of both the `hep-th` and `math.DG` communities. In particular, in six real dimensions these have become known

as *non-Kähler* geometries. They are one of the simplest examples of geometries characterized by a  $G$ -structure, which arise naturally in supergravity theories [1]. From the differential geometry point of view, the interest comes from the fact that the relevant  $SU(3)$  structure is a relatively simple modification of  $SU(3)$  holonomy, the latter characterizing Calabi-Yau three-folds. Essentially one replaces the Levi-Civita connection by a connection with skew-symmetric torsion. The manifolds are then still complex, but not symplectic, hence in particular they are not Kähler manifolds [2, 3]. These structures are, however, quite rich; for example, they are relevant for studies of mirror symmetry [4, 5] and conifold transitions [6, 7]. Other work on heterotic non-Kähler geometries includes, for example, [8]. This is also the appropriate framework for studying supersymmetric configurations of five-branes, and hence it features in the gauge/gravity duality. Most notably, a non-Kähler solution related to  $\mathcal{N} = 1$  SYM theory was discussed in [9]. It is worth emphasizing that there are also generalizations of the non-Kähler equations to other dimensions. For example, in seven and eight dimensions the geometries are characterized by torsion-full  $G_2$  and  $Spin(7)$  structures, respectively [1, 10, 11, 12]. Most of our results apply also to these cases.

In the context of heterotic or type I string theory, there is also a non-Abelian  $SO(32)$  or  $E_8 \times E_8$  gauge field  $\mathcal{A}$ . In the latter case, the non-Abelian gauge field plays a crucial role in constructions of string theory vacua that are designed to reproduce semi-realistic four-dimensional particle physics [13, 14, 15]. Vacua in which the internal field strength has a background expectation value transforming in a subgroup  $G$  of one of the  $E_8$  factors give as four-dimensional gauge group the commutant of  $G$  in  $E_8$ . In the simplest construction, known as the standard embedding, one identifies the internal gauge field with the  $SU(3)$  spin connection of a background Calabi-Yau metric, giving  $E_6$  as the observed gauge group. In the first part of this paper we will mainly focus on a  $U(1)$  gauge field inside the common Abelian subgroup  $U(1)^{16}$  of  $SO(32)$  and  $E_8 \times E_8$ ; we shall then later discuss how to incorporate non-Abelian gauge fields that may lead to four-dimensional gauge groups  $SO(10)$  and  $SU(5)$ , using a modified form of the standard embedding.

The mathematical understanding of non-Kähler geometries is much less developed than that of Calabi-Yau manifolds. Despite the very impressive existence results obtained recently in [16], it is still desirable to construct new explicit examples. The reformulation of the supersymmetry conditions in terms of equations for an  $SU(3)$  structure [2, 3, 17, 10] is in fact not particularly helpful for this task, as one might have hoped. To make progress, the standard approach is that of making an *ansatz* for the metric (or  $G$ -structure) enjoying additional symmetries. In this paper we

will explore a different method for constructing new solutions, namely a *solution-generating* transformation. Generally, these transformations exploit the symmetries of supergravity theories, and here we will discuss how such methods can be adapted to study solutions to the non-Kähler equations, and their cousins in dimensions  $d \neq 6$ .

Buscher T-duality is at the heart of most of the generating techniques in supergravity. A simple transformation involving T-duality is the so-called TsT transformation. This has been applied to a variety of backgrounds, producing interesting gravity duals. The basic idea is that this transformation induces a  $B$ -field on the two-torus on which it is performed, hence giving a non-commutative deformation [18].<sup>1</sup> Examples include the gravity duals of non-commutative field theories [20], the beta-deformations [21], dipole deformations [22], and non-relativistic (Schrödinger) deformations [23]. Essentially the same transformation (see appendix B) was used earlier to construct configurations of D-brane bound states [24]. For a concise review, see [25].

The TsT transformation is part of the  $O(d, d)$  T-duality group<sup>2</sup> of type II supergravities [27, 28]. In the heterotic theory the generalized T-duality group is enlarged to  $O(d+16, d)$ , where 16 is the dimension of the Cartan subgroup of the gauge group [29]. Transformations with elements of this group have been employed in the literature to construct (non-supersymmetric) charged black holes, starting from uncharged black holes [30, 31, 32, 33].

The solution-generating method that we will discuss in this paper, in the context of heterotic supergravity, involves a particular subgroup  $O(2, 1)$  of the  $O(d + 16, d)$  group. However, instead of using the  $O(d + 16, d)$  transformation rules, we will derive the transformation from a rather different point of view, which is analogous to the TsT transformation. Starting from a heterotic solution with a  $U(1)$  symmetry, which includes Calabi-Yau manifolds with a  $U(1)$  isometry as a special case, essentially the transformation that we will employ consists of a rotation of the coordinates on an auxiliary two-torus, followed by an ordinary Buscher T-duality (not involving the gauge fields), and then an opposite rotation. We will refer to this as an rTr transformation. A more detailed description will be given shortly. In particular, we will show that this transformation *preserves supersymmetry*

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<sup>1</sup>Another generating method, that uses a chain of U-dualities, was proposed in [6, 19] and may also be interpreted as a certain non-commutative deformation.

<sup>2</sup>See [26] for an explicit calculation of the  $O(2, 2)$  transformation.

of the heterotic theory,<sup>3</sup> hence in particular it preserves the non-Kähler equations.

In the process we will also show that this procedure may be phrased entirely in terms of a simple  $O(d, d)$  transformation in type II. In fact, we will see that physically the solution-generating method, once embedded into type II, is essentially equivalent to a TsT transformation. This can be applied also to backgrounds which are non-Kähler to begin with. For example, one can apply it to gravity duals of field theories [9]. In this case our formulas then give the supergravity duals of certain dipole deformations, that we briefly discuss at the end of the paper.

For applications to heterotic string theory, the expansion in powers of  $\alpha'$  becomes important. In particular, the Bianchi identity and equations of motion receive corrections which involve the *gravitational* Chern-Simons terms and the full *non-Abelian* heterotic gauge field, already at first order. However, the transformation that we will discuss does not act naturally on these terms. We will nevertheless explain how to restore  $\alpha'$  in our transformation, and combine this with a certain “modified standard embedding”, in order to obtain full solutions to first order in  $\alpha'$ , including a non-trivial non-Abelian gauge field.

The rest of this paper is organized as follows. In section 2 we summarize and review heterotic supergravity. In section 3 we discuss our solution-generating method and present general formulas for the transformation. Section 4 focuses on the non-Kähler deformation of Calabi-Yau geometries. In section 5 we restore  $\alpha'$  and incorporate the heterotic non-Abelian gauge field. In section 6 we discuss our results and possible directions for future work. Appendix A contains a simple computation showing that compatibility of the supersymmetry equations and equations of motion imply that the curvature of the connection used in the modified Bianchi identity is an *instanton*. In appendix B we write general formulas for TsT and TrT transformations (in type II), and note they are (locally) equivalent.

## 2 The low-energy limit of heterotic strings

We begin by reviewing the low-energy limit of heterotic string theory. This is a heterotic supergravity theory with an  $\alpha'$  expansion, where  $1/2\pi\alpha'$  is the heterotic string tension. The reader might think that this is rather standard

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<sup>3</sup>The reader might think that this is guaranteed by T-duality. However, we have been somewhat cavalier in describing the procedure here. The subtleties involved will become clear later in the paper.

material. Although this should be the case, unfortunately there are many misunderstandings in the literature.

The low-energy limit of heterotic string theory is described by a ten-dimensional supergravity theory with metric  $g$ , dilaton  $\Phi$ , three-form  $H$ , and gauge field  $\mathcal{A}$ . For the  $SO(32)$  heterotic theory we may take  $\mathcal{A}$  to be in the fundamental representation; thus locally  $\mathcal{A}$  is a one-form with values in purely imaginary, skew-symmetric  $32 \times 32$  matrices. There is no such representation for the  $E_8 \times E_8$  string, and in general one should replace the particular traces that follow by  $1/30$  of the trace in the adjoint representation. Alternatively, this is equal to the trace in the fundamental representation when restricted to the  $SO(16) \times SO(16)$  subgroup of  $E_8 \times E_8$  – see, for example, [15].

We begin by introducing the covariant derivatives  $\nabla^\pm$  with torsion:

$$(1) \quad \nabla_i^\pm V^j = \nabla_i V^j \pm \frac{1}{2} H_{ik}^j V^k,$$

where  $V$  is any vector field and  $\nabla$  denotes the Levi-Civita connection of  $g$ . These have totally skew-symmetric torsion  $\pm H$ , respectively. The string frame action, up to two loops in sigma model perturbation theory, is<sup>4</sup> [34]

$$(2) \quad S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} e^{-2\Phi} \left[ R + 4(\nabla\Phi)^2 - \frac{1}{12} H_{ijk} H^{ijk} - \alpha' \left( \text{tr } \mathcal{F}_{ij} \mathcal{F}^{ij} - \text{tr } \mathcal{R}_{ij}^- \mathcal{R}^{-ij} \right) \right] + O(\alpha'^2).$$

The corresponding equations of motion are

$$(3) \quad \begin{aligned} R_{ij} + 2\nabla_i \nabla_j \Phi - \frac{1}{4} H_{ikl} H_j{}^{kl} - 2\alpha' \text{tr } \mathcal{F}_{ik} \mathcal{F}_j{}^k \\ + 2\alpha' R_{iklm}^- R_j{}^{klm} + O(\alpha'^2) &= 0, \\ \nabla^2(e^{-2\Phi}) - \frac{1}{6} e^{-2\Phi} H_{ijk} H^{ijk} - \alpha' e^{-2\Phi} \text{tr } \mathcal{F}_{ij} \mathcal{F}^{ij} \\ + \alpha' e^{-2\Phi} \text{tr } \mathcal{R}_{ij}^- \mathcal{R}^{-ij} + O(\alpha'^2) &= 0, \\ \nabla^i (e^{-2\Phi} H_{ijk}) + O(\alpha'^2) &= 0, \\ \nabla^{+i} (e^{-2\Phi} \mathcal{F}_{ij}) + O(\alpha'^2) &= 0. \end{aligned}$$

Here  $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$  is the field strength for  $\mathcal{A}$ , and  $R_{ijkl}^\pm$  denotes the curvature tensor for  $\nabla^\pm$ . We shall denote the corresponding curvature two-forms by  $\mathcal{R}_{ij}^\pm$ , so that

$$(4) \quad \mathcal{R}_{ij}^{\pm ab} = R_{ijkl}^\pm e^{ak} e^{bl},$$

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<sup>4</sup>The convention we use for the  $\alpha'$ -dependent terms differs from some other conventions found in the literature; but different conventions involve only a trivial redefinition of  $\alpha'$ . Note that, as usual in physics, the traces  $\text{tr}$  and generators  $T^a$  of the various non-Abelian groups are defined so that  $\text{tr}(T^a T^b)$  is positive-definite.

where  $e_k^a$  is an orthonormal frame (vielbein). The Bianchi identity for  $H$  is

$$(5) \quad dH = 2\alpha' (\text{tr } \mathcal{F} \wedge \mathcal{F} - \text{tr } \mathcal{R}^- \wedge \mathcal{R}^-) + O(\alpha'^2) .$$

The corresponding supersymmetry equations are [35]

$$(6) \quad \begin{aligned} \nabla_i^+ \epsilon + O(\alpha'^2) &= 0 , \\ \left( \Gamma^i \partial_i \Phi + \frac{1}{12} H_{ijk} \Gamma^{ijk} \right) \epsilon + O(\alpha'^2) &= 0 , \\ \mathcal{F}_{ij} \Gamma^{ij} \epsilon + O(\alpha'^2) &= 0 . \end{aligned}$$

Here  $\epsilon$  is a Majorana-Weyl spinor and  $\Gamma_i$  generate the Clifford algebra  $\{\Gamma_i, \Gamma_j\} = 2g_{ij}$ . We shall refer to the last equation for  $\mathcal{F}$  in (6) as the *instanton equation*. That it is the curvature tensor for  $\nabla^-$  that appears in the Bianchi identity (5), and then correspondingly the equations of motion (3), was first noted by Hull [3], and then subsequently discussed by other authors – see, for example, [36]. This connection has *opposite* sign torsion to the connection  $\nabla^+$  appearing the gravitino equation in (6). Notice that one is free to make a field redefinition  $H \leftrightarrow -H$ , which then exchanges the roles of  $\nabla^\pm$ ; in fact this opposite sign convention is also common in the literature. However, whatever the convention, there is always a *relative* sign difference between the torsion of the connection that appears in the gravitino equation and the connection that appears in the higher derivative terms. A common error in the literature is to confuse these connections. In the mathematics literature the connection  $\nabla^+$  is sometimes referred to as the *Bismut connection*, discussed extensively in the context of the heterotic string, for example, in [39]. We shall refer to  $\nabla^-$  as the *Hull connection*.

Having briefly summarized the low-energy limit of heterotic string theory, in the next section we study a closely-related heterotic supergravity theory. Formally, this theory is obtained by (i) restricting to an Abelian subgroup of the gauge group, (ii) setting the higher order  $O(\alpha'^2)$  corrections to zero in the above theory, and then setting  $\alpha' = 1$ , and (iii) formally setting to zero the higher derivative terms in  $\mathcal{R}^-$ . The solution-generating method we will present in the next section is for this Abelian theory, which we shall call *Abelian heterotic supergravity*.<sup>5</sup> In particular, this theory does not contain  $\alpha'$ . We shall return to the heterotic string, and reinstate  $\alpha'$ , in section 5.

### 3 Heterotic solution-generating method

In this section we present a supersymmetry-preserving solution-generating method in Abelian heterotic supergravity, where the initial solution has a  $U(1)$  symmetry. The construction relies on the observation that Abelian

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<sup>5</sup>The Lagrangian and supersymmetry equations will be given explicitly in section 3.

heterotic supergravity can be embedded supersymmetrically into type II supergravity. An  $r_2 T r_1$  transformation in type II theory, with an appropriate relation between the two rotations  $r_1, r_2$ , is then effectively a solution-generating transformation in heterotic supergravity. We then show that this transformation lies in the  $O(2, 1)$  T-duality group of the heterotic theory itself. In fact the rotation is a close cousin of the Sen transformation, that takes the Kerr black hole to the Kerr-Sen black hole [32]. The latter was recently studied in higher dimensions in [30].

### 3.1 Type II and heterotic supergravities

In this section we begin by showing that Abelian heterotic supergravity may be embedded supersymmetrically into the NS-NS sector of type II supergravity. This expands upon some comments made originally in [10].

The NS-NS sector of type II supergravity is a ten-dimensional theory with metric  $\hat{g}$ , dilaton  $\hat{\Phi}$ , closed three-form  $\hat{H}$ , and string frame Lagrangian

$$(7) \quad \hat{\mathcal{L}} = e^{-2\hat{\Phi}} \left( \hat{R} \hat{\star} 1 + 4 \hat{\star} d\hat{\Phi} \wedge d\hat{\Phi} - \frac{1}{2} \hat{\star} \hat{H} \wedge \hat{H} \right) .$$

The corresponding equations of motion are

$$(8) \quad \begin{aligned} \hat{R}_{ij} + 2\hat{\nabla}_i \hat{\nabla}_j \hat{\Phi} - \frac{1}{4} \hat{H}_{ikl} \hat{H}_j{}^{kl} &= 0 , \\ \hat{\nabla}^2 (e^{-2\hat{\Phi}}) - \frac{1}{6} e^{-2\hat{\Phi}} \hat{H}_{ijk} \hat{H}^{ijk} &= 0 , \\ \hat{\nabla}^i (e^{-2\hat{\Phi}} \hat{H}_{ijk}) &= 0 , \end{aligned}$$

where  $\hat{\nabla}$  denotes the Levi-Civita connection for  $\hat{g}$ . The relevant supersymmetry equations are the gravitino and dilatino equations

$$(9) \quad \begin{aligned} \left( \hat{\nabla}_i \pm \frac{1}{8} \hat{H}_{ijk} \hat{\Gamma}^{jk} \right) \epsilon_{\pm} &= 0 , \\ \left( \hat{\Gamma}^i \partial_i \hat{\Phi} \pm \frac{1}{12} \hat{H}_{ijk} \hat{\Gamma}^{ijk} \right) \epsilon_{\pm} &= 0 . \end{aligned}$$

Here  $\epsilon_{\pm}$  are Majorana-Weyl spinors and  $\hat{\Gamma}_i$  generate the Clifford algebra  $\{\hat{\Gamma}_i, \hat{\Gamma}_j\} = 2\hat{g}_{ij}$ . A type IIA or type IIB solution will be supersymmetric if and only if there is at least one  $\epsilon_+$  or  $\epsilon_-$  satisfying (9), where  $\epsilon_{\pm}$  have the opposite or same chirality, respectively. For the application we have in mind here we are interested in solutions with a single spinor  $\epsilon_+$ , which without essential loss of generality we take to have positive chirality.

Consider such a supersymmetric solution which is invariant under a Killing vector field  $\partial_z$  which is nowhere zero. In particular, this should preserve the

spinor  $\epsilon_+$  under the spinor Lie derivative, as well as  $\hat{\Phi}$  and  $\hat{H}$ . We may in general then write

$$\begin{aligned} \hat{g} &= e^{2\varphi}(dz + A_2)^2 + g, \\ \hat{\Phi} &= \Phi + \frac{1}{2}\varphi, \\ \hat{H} &= H + F_1 \wedge (dz + A_2). \end{aligned} \quad (10)$$

Here we have nine-dimensional fields, transverse to and invariant under  $\partial_z$ , comprising a metric  $g$ , scalar fields  $\Phi$ ,  $\varphi$ , Abelian gauge fields  $A_1$ ,  $A_2$  with curvatures  $F_1 = dA_1$ ,  $F_2 = dA_2$ , and a three-form  $H$ . These satisfy equations of motion derived from the nine-dimensional Lagrangian

$$\begin{aligned} \mathcal{L} &= e^{-2\Phi} \left( R \star 1 + 4 \star d\Phi \wedge d\Phi - \star d\varphi \wedge d\varphi \right. \\ &\quad \left. - \frac{1}{2} e^{-2\varphi} \star F_1 \wedge F_1 - \frac{1}{2} e^{2\varphi} \star F_2 \wedge F_2 - \frac{1}{2} \star H \wedge H \right), \end{aligned} \quad (11)$$

while the original Bianchi identity  $d\hat{H} = 0$  becomes  $dH = -F_1 \wedge F_2$ . The supersymmetry equations (9) correspondingly reduce to equations for a nine-dimensional spinor  $\psi$  satisfying

$$\begin{aligned} \left( \nabla_\alpha + \frac{1}{8} H_{\alpha\beta\gamma} \sigma^{\beta\gamma} \right) \psi &= \frac{i}{4} (e^{-\varphi} F_{1\alpha\beta} + e^{\varphi} F_{2\alpha\beta}) \sigma^\beta \psi, \\ \left( \sigma^\alpha \partial_\alpha \Phi + \frac{1}{12} H_{\alpha\beta\gamma} \sigma^{\alpha\beta\gamma} \right) \psi &= \frac{i}{8} (e^{-\varphi} F_{1\alpha\beta} + e^{\varphi} F_{2\alpha\beta}) \sigma^{\alpha\beta} \psi, \\ \partial_\alpha \varphi \sigma^\alpha \psi &= \frac{i}{4} (e^{-\varphi} F_{1\alpha\beta} - e^{\varphi} F_{2\alpha\beta}) \sigma^{\alpha\beta} \psi. \end{aligned} \quad (12)$$

In deriving these equations we have used the fact that the ten-dimensional Killing spinor is invariant,  $\mathcal{L}_{\partial/\partial z} \epsilon = 0$ . Here  $\sigma_\alpha$  generate the Clifford algebra  $\{\sigma_\alpha, \sigma_\beta\} = 2g_{\alpha\beta}$ , and we note that it is possible to choose conventions such that the Majorana condition in ten dimensions reduces to  $\psi = \psi^*$ , with the  $\sigma_\alpha$  being symmetric and purely imaginary. We have taken a judicious linear combination of the reduced dilatino equation and  $z$ -component of the gravitino equation in presenting (12).

A T-duality in ten dimensions along  $\partial_z$  is equivalent to a  $\mathbb{Z}_2$  transformation of the nine-dimensional theory in which  $\varphi \leftrightarrow -\varphi$ ,  $A_1 \leftrightarrow A_2$ , with the other fields being invariant. In particular, notice that the supersymmetry equations (12) are manifestly invariant, with the last equation changing sign. This is a simple, direct proof that T-duality along such a direction preserves supersymmetry.

We also notice that a solution with  $\varphi = 0$  and  $A_2 = \mathcal{A} = -A_1$  has equations of motion that may be derived from the Lagrangian

$$(13) \quad \mathcal{L}_{\text{Het}} = e^{-2\Phi} \left( R \star 1 + 4 \star d\Phi \wedge d\Phi - \star \mathcal{F} \wedge \mathcal{F} - \frac{1}{2} \star H \wedge H \right),$$



namely

$$\begin{aligned}
R_{\alpha\beta} + 2\nabla_\alpha \nabla_\beta \Phi - \frac{1}{4} H_{\alpha\gamma\delta} H_{\beta}^{\gamma\delta} - \mathcal{F}_{\alpha\gamma} \mathcal{F}_{\beta}^{\gamma} &= 0 , \\
\nabla^2(e^{-2\Phi}) - \frac{1}{6} e^{-2\Phi} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} - \frac{1}{2} e^{-2\Phi} \mathcal{F}_{\alpha\beta} \mathcal{F}^{\alpha\beta} &= 0 , \\
\nabla^\alpha (e^{-2\Phi} H_{\alpha\beta\gamma}) &= 0 , \\
\nabla^\alpha (e^{-2\Phi} \mathcal{F}_{\alpha\beta}) - \frac{1}{2} e^{-2\Phi} \mathcal{F}^{\gamma\delta} H_{\beta\gamma\delta} &= 0 .
\end{aligned}
\tag{14}$$

Here  $\mathcal{F} = d\mathcal{A}$ , the Bianchi identity is  $dH = \mathcal{F} \wedge \mathcal{F}$ , and the supersymmetry equations are

$$\begin{aligned}
\left( \nabla_\alpha + \frac{1}{8} H_{\alpha\beta\gamma} \sigma^{\beta\gamma} \right) \psi &= 0 , \\
\left( \sigma^\alpha \partial_\alpha \Phi + \frac{1}{12} H_{\alpha\beta\gamma} \sigma^{\alpha\beta\gamma} \right) \psi &= 0 , \\
\mathcal{F}_{\alpha\beta} \sigma^{\alpha\beta} \psi &= 0 .
\end{aligned}
\tag{15}$$

These are the supersymmetry equations for a nine-dimensional heterotic supergravity theory, with Abelian gauge field  $\mathcal{A}$  and curvature  $\mathcal{F} = d\mathcal{A}$ . More precisely, of course *Abelian heterotic supergravity* exists in ten dimensions. This has the same Lagrangian, equations of motion and supersymmetry equations as (13), (14) and (15), respectively, while in the present context the latter arise as a *trivial* reduction of the ten-dimensional theory to nine dimensions; that is, the ten-dimensional solution is assumed to be invariant under a *constant length* Killing vector field, with all fields being transverse to (no “legs”) and invariant under this direction. In particular, any supersymmetric solution to the above nine-dimensional theory trivially lifts to a supersymmetric ten-dimensional Abelian heterotic solution.

We also see that any supersymmetric nine-dimensional heterotic solution lifts to a supersymmetric type II solution, via the ansatz

$$\begin{aligned}
\hat{g} &= (dz + \mathcal{A})^2 + g , \\
\hat{\Phi} &= \Phi , \\
\hat{H} &= H - \mathcal{F} \wedge (dz + \mathcal{A}) .
\end{aligned}
\tag{16}$$

It is in this sense that (nine-dimensional) Abelian heterotic supergravity is embedded into type II supergravity.

### 3.2 The solution-generating rotation

The idea in this section is to combine the observation of the previous section with certain standard solution-generating methods in type II. Since Abelian heterotic supergravity embeds into type II supergravity, it follows that the corresponding duality symmetries will also be related.

We begin with a supersymmetric Abelian heterotic solution embedded into type II, via (16). In addition we assume that the heterotic solution has a  $U(1)$  symmetry that preserves the Killing spinor  $\psi$ . We may thus write the heterotic solution as

$$\begin{aligned} g &= \frac{1}{V}(\mathrm{d}\tau + A)^2 + V g_{\perp} , \\ \mathcal{A} &= a \mathrm{d}\tau + \mathcal{A}_{\perp} , \\ B &= B_1 \wedge \mathrm{d}\tau + B_{\perp} , \end{aligned} \tag{17}$$

where we have introduced the  $B$ -field via

$$H = \mathrm{d}B + \mathcal{A} \wedge \mathrm{d}\mathcal{A} . \tag{18}$$

Here  $\partial_{\tau}$  is the Killing vector field that generates the  $U(1)$  symmetry.  $V$  and  $a$  are functions,  $\mathcal{A}_{\perp}$ ,  $B_1$  are one-forms, and  $B_{\perp}$  is a two-form, all of which are invariant under  $\partial_{\tau}$  and transverse to it.<sup>6</sup> Notice that although  $\mathcal{A}$  transforms under Abelian gauge transformations,  $\tau$ -independent gauge transformations leave  $a$  gauge invariant.

The equations in (16) embed such a solution into type II supergravity. The corresponding type II solution now has *two* symmetries, generated by  $\partial_z$  and  $\partial_{\tau}$ . We may thus apply an  $r_2 T r_1$  transformation. More precisely, we first rotate  $(z, \tau)$  by an  $SO(2) \cong U(1)$  rotation with constant angle  $\delta_1 \in [0, 2\pi)$ , then perform a T-duality along the new first coordinate  $\tilde{z}$ , and finally perform another rotation with constant angle  $\delta_2 \in [0, 2\pi)$ . These operations manifestly preserve supersymmetry in the type II theory. Having done this, we may then reduce back to nine dimensions on the final  $z$  circle, to obtain a supersymmetric nine-dimensional solution to the theory with Lagrangian (11). We denote these  $r_2 T r_1$ -transformed fields with a prime.<sup>7</sup>

In general such transformations do not preserve the original embedding of the heterotic theory into type II. However, an explicit calculation shows that the embedding is indeed preserved provided one takes  $\delta_1 = -\delta_2 = \delta$ . In particular, with this choice the final nine-dimensional rotated gauge fields  $A'_1, A'_2$  for the theory with Lagrangian (11) obey  $A'_2 = -A'_1$ , while  $\varphi' = 0$ . We may denote this transformation more precisely as  $r^{-1} T r$ , and the above procedure then leads to a supersymmetric heterotic solution given explicitly

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<sup>6</sup>That is, they are *basic* forms with respect to  $\partial_{\tau}$ .

<sup>7</sup>We note that essentially this procedure was applied to higher-dimensional black hole solutions in [30], with the rotations replaced by boosts so that  $\partial_{\tau}$  is a timelike direction.

by

$$\begin{aligned}
g' &= \frac{1}{V'} (d\tau' + A')^2 + V g_\perp , \\
e^{2\Phi'} &= \frac{1}{h} e^{2\Phi} , \\
\mathcal{A}' &= a' d\tau' + \mathcal{A}'_\perp , \\
(19) \quad B' &= B'_1 \wedge d\tau' + B'_\perp ,
\end{aligned}$$

where we have defined the functions

$$\begin{aligned}
h &\equiv (c - sa)^2 + \frac{s^2}{V} , \\
(20) \quad V' &\equiv V h^2 = V \left[ (c - sa)^2 + \frac{s^2}{V} \right]^2 ,
\end{aligned}$$

and  $c = \cos \delta$ ,  $s = \sin \delta$ . The rotated fields are

$$\begin{aligned}
a' &= -\frac{1}{h} \left[ (c^2 - s^2)a + cs \left( 1 - a^2 - \frac{1}{V} \right) \right] , \\
A' &= (c - sa)^2 A + s(c - sa) \mathcal{A}_\perp + s(c \mathcal{A}_\perp + s B_1) , \\
\mathcal{A}'_\perp &= -\frac{1}{h} \left[ (c - sa)(c \mathcal{A}_\perp + s B_1) - \frac{s}{V} (s \mathcal{A}_\perp + (c - sa)A) \right] , \\
B'_1 &= \frac{1}{h} \left[ (c - sa)(c B_1 - s \mathcal{A}_\perp) - \frac{s}{V} (c \mathcal{A}_\perp - (s + ca)A) \right] , \\
(21) \quad B'_\perp &= B_\perp + \frac{s}{h} \left[ (c - sa) B_1 \wedge \mathcal{A}_\perp - \frac{1}{V} (c \mathcal{A}_\perp + s B_1) \wedge A \right] .
\end{aligned}$$

It is not surprising that this procedure can be understood entirely within the original heterotic theory itself. Our initial heterotic solution (17) has a  $U(1)$  symmetry generated by  $\partial_\tau$  and a single Abelian gauge field  $\mathcal{A}$ , and in fact the above rotation is then embedded into the  $O(2,1)$  duality group of this theory. Similar transformations were first investigated by Hassan and Sen [29], [32]. The above procedure effectively “geometrizes” this  $O(2,1)$  transformation as an  $O(2,2)$  transformation of type II, now with two symmetries generated by  $\partial_z$ ,  $\partial_\tau$ . As well as giving a first principles proof that the transformation preserves supersymmetry, the above construction also embeds it into type II theory, which we shall elaborate on later in section 6. We note that in reference [40] it was shown quite generally that such duality transformations preserve *worldsheet* supersymmetry.

Let us see explicitly that (19) indeed lies in the  $O(2,1)$  duality group of the heterotic theory, following [27] and [28]. We begin by making the

change of variables

$$(22) \quad \begin{aligned} \mathcal{V} &= \sqrt{2}(\mathcal{A}_\perp - aA) , \\ C &= -B_1 + a\mathcal{A}_\perp - a^2A . \end{aligned}$$

We then form a triplet of Abelian gauge fields, defining

$$(23) \quad \mathbf{A} = \begin{pmatrix} A \\ C \\ \mathcal{V} \end{pmatrix} .$$

It is then straightforward to check that the rotation (19) acts on  $\mathbf{A}$  as the  $O(2,1)$  matrix

$$(24) \quad \mathcal{R} = \begin{pmatrix} \cos^2 \delta & -\sin^2 \delta & \frac{1}{\sqrt{2}} \sin 2\delta \\ -\sin^2 \delta & \cos^2 \delta & \frac{1}{\sqrt{2}} \sin 2\delta \\ \frac{1}{\sqrt{2}} \sin 2\delta & \frac{1}{\sqrt{2}} \sin 2\delta & -\cos 2\delta \end{pmatrix} .$$

Here the metric  $\eta$  preserved by  $\mathcal{R}$  is

$$(25) \quad \eta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

so that  $\mathcal{R}^T \eta \mathcal{R} = \eta$ . Notice that  $\mathcal{R}$  is symmetric,  $\mathcal{R}^T = \mathcal{R}$ , and also that  $\mathcal{R}^2 = 1$ . This latter fact is consistent with the original construction of the duality transformation in type II, where  $\mathcal{R} = r^{-1} T_{\text{Type II}} r$ . We also note that  $\det \mathcal{R} = -1$ , and hence the transformation is *not* continuously connected to the identity, in general.<sup>8</sup> In particular, notice that

$$(26) \quad \begin{aligned} \mathcal{R}_{\delta=0} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \\ \mathcal{R}_{\delta=\pi/2} &= \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = T_{\text{Het}} . \end{aligned}$$

Thus for  $\delta = 0$  the transformed solution differs from the original solution by reversing the sign of the Abelian gauge field  $\mathcal{F}$ . Note this is an obvious discrete symmetry of heterotic supergravity. We have also noted that  $\mathcal{R}_{\delta=\pi/2}$

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<sup>8</sup>In the special case in which one begins with a Calabi-Yau solution with zero gauge field, it *is* connected to the identity. We discuss this further in the next section.

is precisely *heterotic T-duality*. In terms of the original variables, this is

$$\begin{aligned}
 g' &= \frac{1}{V(a^2 + \frac{1}{V})^2} [d\tau' + B_1 - a(\mathcal{A}_\perp - aA)]^2 + Vg_\perp, \\
 e^{2\Phi'} &= \frac{1}{(a^2 + \frac{1}{V})} e^{2\Phi}, \\
 \mathcal{A}' &= \frac{1}{(a^2 + \frac{1}{V})} \left[ a(d\tau' + B_1) + \frac{1}{V}(\mathcal{A}_\perp - aA) \right], \\
 (27) \quad B' &= B_\perp - \frac{1}{(a^2 + \frac{1}{V})} (d\tau' + B_1) \wedge \left( a\mathcal{A}_\perp + \frac{1}{V}A \right).
 \end{aligned}$$

These are the heterotic Buscher T-duality rules of [28].

Next we turn to the transformation of the scalars. Again following [27] and [28], we define a  $3 \times 3$  matrix  $M$  via

$$(28) \quad M^{-1} = \begin{pmatrix} \frac{1}{V} (1 + a^2 V)^2 & -a^2 V & \sqrt{2}a (1 + a^2 V) \\ -a^2 V & V & -\sqrt{2}a V \\ \sqrt{2}a (1 + a^2 V) & -\sqrt{2}a V & 1 + 2a^2 V \end{pmatrix}.$$

This is also symmetric and lies in  $SO(2, 1)$ , namely  $(M^{-1})^T \eta M^{-1} = \eta$  and  $\det M = 1$ . It is straightforward to check that the rotation formulae (20), (21) are equivalent to

$$(29) \quad (M')^{-1} = \mathcal{R} M^{-1} \mathcal{R}^T.$$

Finally, one can check that the two-form

$$(30) \quad \mathcal{B} = B_\perp - \frac{1}{2} B_1 \wedge A + \frac{1}{2} a A \wedge \mathcal{A}_\perp$$

is invariant. One can write a reduced action, where one reduces along  $\partial_\tau$ , which is manifestly  $O(2, 1)$ -invariant [27], [28]. Regarding the initial heterotic solution as ten-dimensional, in the above variables this takes the form

$$\begin{aligned}
 S &= \int d^9 x \sqrt{-g} e^{-2\phi} \left[ R + 4(d\phi)^2 - \frac{1}{2} H^2 + \frac{1}{8} \text{tr} (\partial_\mu M^{-1} \partial^\mu M) \right. \\
 (31) \quad &\quad \left. - \frac{1}{2} \mathbf{F}^T M^{-1} \mathbf{F} \right]
 \end{aligned}$$

where  $\phi = \Phi + \frac{1}{4} \log V$  and

$$(32) \quad H = d\mathcal{B} + \frac{1}{2} \mathbf{A}^T \eta \wedge \mathbf{F}$$

is  $O(2, 1)$  invariant, and we have defined  $\alpha^2 \equiv \frac{1}{p!} \alpha_{i_1 \dots i_p} \alpha^{i_1 \dots i_p}$  for a  $p$ -form  $\alpha$ .

### 3.3 The Calabi-Yau case

In this subsection we specialize to the case where the initial heterotic solution has  $B = \mathcal{A} = 0$  and  $\Phi = \Phi_0$  is constant. In particular, this means that the initial metric  $g$  has special holonomy, *e.g.* a product of Minkowski space with a Calabi-Yau manifold or a  $G_2$  holonomy manifold.

In this case the rotation (19) simplifies to

$$\begin{aligned}
 g' &= \frac{1}{V'}(d\tau' + \cos^2 \delta A)^2 + Vg_\perp , \\
 e^{2\Phi'} &= \frac{1}{h} e^{2\Phi_0} , \\
 \mathcal{A}' &= -\frac{\sin \delta \cos \delta}{h} \left[ d\tau' - \frac{1}{V}(d\tau' + A) \right] , \\
 B' &= -\frac{h \sin^2 \delta}{V'} d\tau' \wedge A ,
 \end{aligned}
 \tag{33}$$

where as before  $V' = Vh^2$ , but now  $h$  simplifies to

$$h = \cos^2 \delta + \frac{\sin^2 \delta}{V} .
 \tag{34}$$

Recall that setting  $\delta = \pi/2$  simply gives the T-dual heterotic solution. Provided  $\delta \neq \pi/2$  we may introduce the new coordinate  $w = \tau' / \cos^2 \delta$ , and rewrite (33) slightly as

$$\begin{aligned}
 g' &= \frac{\cos^4 \delta}{V'}(dw + A)^2 + Vg_\perp , \\
 e^{2\Phi'} &= \frac{1}{h} e^{2\Phi_0} , \\
 \mathcal{A}' &= -\sin \delta \cos \delta \left[ dw - \frac{h}{V'}(dw + A) \right] , \\
 B' &= -\frac{h \sin^2 \delta \cos^2 \delta}{V'} dw \wedge A .
 \end{aligned}
 \tag{35}$$

The curvatures are

$$\begin{aligned}
 \mathcal{F}' &= \sin \delta \cos \delta d \left[ \frac{h}{V'}(dw + A) \right] , \\
 H' &= \frac{h^2 \sin^2 \delta \cos^2 \delta}{V'^2} (dw + A) \wedge F .
 \end{aligned}
 \tag{36}$$

Notice that a very special case of this construction is obtained by starting with flat ten-dimensional Minkowski spacetime. Even in this case our transformation produces a non-Kähler geometry with non-trivial  $B$  and gauge

field. It would be interesting to study whether one can quantize strings in these backgrounds.

### 3.4 Global analysis

The heterotic rotation we have described gives by construction a new set of fields which satisfy the supersymmetry equations, Bianchi identity and equations of motion. One can then ask whether the rotated solution is a *regular* supergravity solution. In the first part of this subsection we address this question for the simplified case in section 3.3. We prove that for all  $\delta \in [0, \pi/2)$  the rotated solution is a smooth supergravity solution, with the underlying manifold remaining the same. In this sense, the deformation is like the beta deformation of Maldacena-Lunin [21], in that the parameter  $\delta$  may be varied continuously to give a smoothly connected one-parameter family of supergravity solutions. We then discuss the more general rotation of section 3.2. As for the beta deformation of [21] a general global analysis is more complicated, and we restrict ourselves here only to some brief comments on global issues in this more general setting.

We begin with (35), which is valid for  $\delta \neq \pi/2$ . Consider first the locus of points  $\mathcal{M}_0 \subset \mathcal{M}$  in the original spacetime  $\mathcal{M}$  where  $\partial_\tau$  is non-zero. Since we assumed that  $\partial_\tau$  generates a  $U(1)$  isometry, it follows that  $\mathcal{M}_0$  will fibre over some orbifold  $\mathcal{M}_0/U(1)$ . The one-form  $(d\tau + A)$  is then a connection one-form on the corresponding  $U(1)$  principal (orbi-)bundle. Notice that on  $\mathcal{M}_0$  the function  $V$  is finite and strictly positive, and from (34) the same is true of  $h$ , and thus also of  $V' = Vh^2$ . Since the one-form  $(dw + A)$  also appears in the rotated metric in (35), it follows that the corresponding locus  $\mathcal{M}'_0 \subset \mathcal{M}'$ , where  $\partial_w$  is non-zero in the rotated spacetime  $\mathcal{M}'$ , is diffeomorphic to  $\mathcal{M}_0$ , and that the rotated metric is regular on  $\mathcal{M}'_0 \cong \mathcal{M}_0$  provided one takes the period of  $w$  to be the *same* as the period of  $\tau$ . The rotation then simply rescales the size of the  $U(1)$  fibre.

Let us now look at the locus of points where  $\partial_\tau$  vanishes. Take some connected component  $S \subset \mathcal{M}$ . By definition,  $V$  will diverge along  $S$  – the way in which it diverges is crucial for the regularity of the metric in a neighbourhood of  $S$ . Essentially, the metric restricted to the normal directions to  $S$  should approach the flat space metric in some polar coordinate system. It turns out we will not need to enter into the precise details of this metric regularity condition. To see this, notice that near to  $S$  we have to leading order  $h \sim \cos^2 \delta$ , and  $V' \sim \cos^4 \delta V$ . Thus near to  $S$  the rotated metric is to

leading order

$$(37) \quad g' \approx \frac{1}{V}(dw + A)^2 + Vg_{\perp}.$$

This is *identical* to the initial metric, with  $\tau$  replaced by the new coordinate  $w$  (which forms the diffeomorphism of  $\mathcal{M}_0$  onto  $\mathcal{M}'_0$ ). Since these have the same period, and since the metrics are the same to leading order near to  $S$ , it follows that the rotated metric will be regular at  $S$  provided the initial metric is. In fact an alternative way to argue this is to note quite generally that

$$(38) \quad g' = g - \frac{\sin^2 \delta (\sin^2 \delta + 2V \cos^2 \delta)}{V (\sin^2 \delta + V \cos^2 \delta)^2} (dw + A)^2.$$

Then where  $V \rightarrow 0$  we have to leading order

$$(39) \quad g' \approx g - \frac{\sin^2 \delta}{V} (dw + A)^2.$$

The second term is a global one-form on  $\mathcal{M}'$ , as discussed in more detail below, and hence the rotated metric is regular at these loci. Also notice that since  $e^{2\Phi'} = \frac{1}{h} e^{2\Phi_0}$ , and  $h$  is smooth and nowhere zero, the rotated dilaton is smooth.

We conclude that for all  $\delta \in [0, \pi/2)$  the underlying manifold is the same,  $\mathcal{M}' \cong \mathcal{M}$ , and the rotated metric is smooth provided the initial metric was. Notice that, in general, this is not true for the T-duality limit  $\delta = \pi/2$ . For example, a codimension four fixed point set  $S$  of  $\partial_{\tau}$  becomes a five-brane in the T-dual solution. In this limit  $h = \frac{1}{V}$  and  $V' = \frac{1}{V}$ . In particular, along  $S$  the dilaton  $e^{2\Phi'} = \frac{1}{h} e^{2\Phi_0} = V e^{2\Phi_0}$  diverges to infinity, as expected for a five-brane supergravity solution. Of course, in this case the T-dual solution has no gauge field:  $\mathcal{A}' = 0$ .

Let us now turn to the rotated curvatures in (36). First note that  $(dw + A)$  is a global smooth one-form on  $\mathcal{M}'_0 \cong \mathcal{M}_0$ . The one-form  $\frac{h}{V'}(dw + A)$  is then a global smooth one-form on the whole spacetime  $\mathcal{M}' \cong \mathcal{M}$ , since  $\frac{h}{V'}$  smoothly tends to zero precisely along the loci where  $\partial_w$  vanishes. More precisely,  $\frac{h}{V'}$  vanishes as  $r^2$ , where  $r$  is the distance to  $S$ , while  $(dw + A)$  approaches a global one-form on the normal sphere bundle to  $S$  in  $M$ . Thus  $\mathcal{F}'$  is actually an *exact* two-form. There is hence no quantized flux associated to the rotated gauge field, as one might expect given that  $\delta$  may be turned on continuously. Similar comments apply also to  $H'$ , showing that it is a smooth global three-form on  $\mathcal{M}'$ . Notice that in the T-dual limit at  $\delta = \pi/2$ , the change in the global structure of the spacetime due to a five-brane would then similarly be accompanied by a delta function source in the Bianchi identity for  $H'$ .



This completes our discussion of the global regularity of the solutions in section 3.3. We conclude this section by commenting briefly on the more general rotations in (19). Again, for  $\delta \neq \pi/2$  one can replace  $\tau' = w \cos^2 \delta$ , and similar analysis shows that the new connection one-form on  $\mathcal{M}'_0$  is now (see (22))

$$(40) \quad dw + A + \sqrt{2} \tan \delta \mathcal{V} - \tan^2 \delta C .$$

The (orbi)-bundle  $\mathcal{M}'_0 \rightarrow \mathcal{M}'_0/U(1)$  will then have the same topology as  $\mathcal{M}_0 \rightarrow \mathcal{M}_0/U(1)$ , for all values of  $\delta \in [0, \pi/2)$ , only if  $\mathcal{V} = \sqrt{2}(\mathcal{A}_\perp - aA)$  and  $C$  (equivalently  $B_1$ ) are *global* one-forms on  $\mathcal{M}_0/U(1)$ . *A priori*, this need not be the case. One can similarly analyse the metric near to  $S$ . In this case the rotated metric is to leading order

$$(41) \quad g' \approx \frac{1}{V} (dw + A + 2 \tan \delta \mathcal{A}_\perp)^2 + V g_\perp .$$

Crucial here is that, as well as  $1/V$  going to zero along  $S$ , one also has that  $a$  and  $B_1$  tend to zero along  $S$ . This is because near to  $S$  the coordinate  $\tau$  is an angular coordinate on the sphere linking  $S$  in  $\mathcal{M}$ , so that  $d\tau$  is in fact singular at  $S$ . It follows that the coefficient of  $d\tau$  in any smooth form must go to zero along  $S$ , at an appropriate rate. The above formula (41) then guarantees that the rotated metric is smooth at  $S$  if the initial metric is.

#### 4 Non-Kähler geometries

As first derived in [2, 3], a supersymmetric heterotic supergravity solution of the Poincaré-invariant form  $\mathbb{R}^{1,3} \times X$  implies that the six-manifold  $X$  has a canonical  $SU(3)$  structure, satisfying the system of equations

$$(42) \quad d(e^{-2\Phi} \Omega) = 0 ,$$

$$(43) \quad d(e^{-2\Phi} \omega \wedge \omega) = 0 ,$$

$$(44) \quad i(\bar{\partial} - \partial)\omega = H .$$

Here  $\omega$  and  $\Omega$  are the  $SU(3)$ -invariant real two-form and complex three-form, respectively. In particular, equation (42) implies that the associated almost complex structure is integrable, so that  $X$  is a complex manifold with zero first Chern class. The  $\partial$  and  $\bar{\partial}$  operators in (44) are then the usual Dolbeault operators. On the other hand, in general  $\omega$  is not closed and hence  $X$  is not a Kähler manifold. Such structures are now commonly referred to as *non-Kähler geometries*, even though this nomenclature is somewhat vague. We note that very similar results hold for products of Minkowski space with complex  $n$ -folds, and also for other  $G$  structures, such as  $G_2$ -structure manifolds [41, 1, 10]. Our analysis in this section should extend

appropriately to all of these cases, but we content ourselves here with the most interesting case of an  $SU(3)$  structure.

For Abelian heterotic supergravity, with no  $\alpha'$ , the supersymmetry equations (42), (43), (44) are also supplemented by the gauge field equations

$$(45) \quad \mathcal{F} \wedge \Omega = 0 ,$$

$$(46) \quad \omega \lrcorner \mathcal{F} = 0 .$$

The first equation (45) says that the  $U(1)$  gauge field strength  $\mathcal{F}$  has Hodge type  $(1,1)$ , while (46) is the *Hermitian-Yangs-Mills* (HYM) equation for the gauge field. These are equivalent to the instanton equation for  $\mathcal{F}$ . The Bianchi identity is

$$(47) \quad dH = \mathcal{F} \wedge \mathcal{F} .$$

In particular, the latter then ensures that the equations of motion are satisfied (see also appendix A).

#### 4.1 The rotated non-Kähler equations

Starting with a supersymmetric  $\mathbb{R}^{1,3} \times X$  heterotic supergravity solution in ten dimensions, the  $O(2,1)$  transformation of section 3 produces a new  $\mathbb{R}^{1,3} \times X'$  solution. If  $X$  is a Calabi-Yau three-fold, then we have already shown in general that provided  $\delta \in [0, \pi/2)$  then  $X'$  is diffeomorphic to  $X$ . However, for  $\delta \neq 0$   $X'$  is equipped with a new  $SU(3)$  structure, which is non-Kähler. In this section we write explicit formulas for the rotated  $SU(3)$  structure, and briefly demonstrate how the  $SU(3)$  structure equations, HYM equation and Bianchi identity are satisfied. We will be particularly interested in the transformation of the holomorphic  $(3,0)$ -form  $\Omega$  in the next subsection.

We now suppress the Minkowski space directions, which may be absorbed into the rotation-invariant  $Vg_\perp$  and play no role, and denote the initial Calabi-Yau three-fold metric by

$$(48) \quad g = \frac{1}{V} (d\tau + A)^2 + Vg_\perp ,$$

where  $g_\perp$  is now a five-dimensional transverse metric. The initial Kähler form and holomorphic  $(3,0)$ -form are, in a hopefully obvious notation,

$$(49) \quad \begin{aligned} \omega &= e_\perp \wedge (d\tau + A) + V\omega_\perp , \\ \Omega &= (Ve_\perp + i(d\tau + A)) \wedge V^{1/2}\Omega_\perp . \end{aligned}$$

Since the initial Killing spinor is assumed to be invariant under  $\partial_\tau$ , it follows that so are  $\omega$  and  $\Omega$ . Notice that  $Ve_\perp = J(d\tau + A)$ , where the complex structure, metric and Kähler form are related via  $J^i_j = g^{ik}\omega_{kj}$ . With this

sign convention,  $d^c f \equiv Jdf = i(\partial - \bar{\partial})f$  acting on functions  $f$ . This sign convention is opposite to much of the mathematics literature, but is more common in the physics literature.

The closure of  $\omega$  and  $\Omega$  immediately lead to

$$(50) \quad de_{\perp} = 0, \quad d(V^{1/2}\Omega_{\perp}) = 0.$$

Denoting rotated quantities with primes as before, we have

$$(51) \quad e^{-2\Phi'}\Omega' = \cos^2 \delta \Omega + \sin^2 \delta e_{\perp} \wedge V^{1/2}\Omega_{\perp},$$

which we immediately see is closed, as required by (42). In these equations one should understand that we have formally replaced  $\tau$  by  $w$  in unrotated quantities such as  $\Omega$  – recall this is the diffeomorphism between  $X$  and  $X'$ . Using the explicit form of  $\mathcal{F}'$  in (36) it is also straightforward to check that  $\mathcal{F}' \wedge \Omega' = 0$ , so that  $\mathcal{F}'$  is type  $(1, 1)$ .

The rotated two-form is

$$(52) \quad e^{-2\Phi'}\omega' = \cos^2 \delta \omega + \sin^2 \delta \omega_{\perp},$$

so that

$$(53) \quad e^{-2\Phi'}\omega' \wedge \omega' = \cos^2 \delta \omega \wedge \omega + \sin^2 \delta V\omega_{\perp} \wedge \omega_{\perp}.$$

Hence

$$(54) \quad d(e^{-2\Phi'}\omega' \wedge \omega') = 0 \quad \Leftrightarrow \quad d(V\omega_{\perp} \wedge \omega_{\perp}) = 0.$$

But this follows immediately using (50) and

$$(55) \quad \omega_{\perp} \wedge \omega_{\perp} = \frac{1}{2}\Omega_{\perp} \wedge \bar{\Omega}_{\perp},$$

which in turns follows from  $\frac{1}{3!}\omega^3 = \frac{i}{8}\Omega \wedge \bar{\Omega}$ .

Next we compute

$$(56) \quad \begin{aligned} e^{-4\Phi'}\omega' \wedge \omega' \wedge \mathcal{F}' &= \sin \delta \cos^3 \delta \omega_{\perp} \wedge (dw + A) \wedge \left[ dV \wedge \omega_{\perp} \right. \\ &\quad \left. - 2e_{\perp} \wedge F \right]. \end{aligned}$$

The right hand side of (56) vanishes on using

$$(57) \quad d\omega = 0 \quad \Leftrightarrow \quad d(V\omega_{\perp}) = e_{\perp} \wedge F,$$

together with (54), which thus establishes the HYM equation (46).

Finally, we compute

$$(58) \quad \begin{aligned} d\omega' &= d(\omega - \sin^2 \delta V'^{-1}e_{\perp} \wedge (dw + A)) , \\ &= \tan \delta e_{\perp} \wedge \mathcal{F}' . \end{aligned}$$

For a  $(1,1)$ -form  $\omega$  we have  $d^c\omega = J \circ d\omega$ , since  $J \circ \omega = \omega$ . Using also the fact that  $\mathcal{F}'$  is type  $(1,1)$  with respect to the transformed complex structure, it follows that

$$\begin{aligned}
 d^c\omega' &= J' \circ d\omega' = \tan \delta J'(e_\perp) \wedge \mathcal{F}' , \\
 &= -\frac{h^2 \sin^2 \delta \cos^2 \delta}{V'^2} (dw + A) \wedge F , \\
 (59) \qquad &= -H' .
 \end{aligned}$$

Finally, in our conventions  $d^c\omega' = i(\partial - \bar{\partial})\omega'$ , so we obtain

$$(60) \qquad H' = i(\bar{\partial} - \partial)\omega' ,$$

which is equation (44).

The above provides an alternative proof that the rotation preserves supersymmetry in this particular case. As already mentioned, very similar computations could be done starting with other special holonomy manifolds. However, the main reason for presenting the above results is that we shall investigate in more detail how the complex structure and other pure spinor (in the sense of [42])  $e^{i\omega}$  transform in the next section.

## 4.2 Invariance of the complex structure

In this section we prove that the rotated complex structure, starting with a Calabi-Yau three-fold, is in fact invariant. Equivalently,  $\Omega'$  is proportional to the original  $\Omega$ , after an appropriate diffeomorphism. This claim is not at all obvious from the formula (51) for  $\Omega'$  in terms of  $\Omega$ . Thus the heterotic rotation may be regarded as fixing the underlying complex manifold, but rotating the solution from Kähler to non-Kähler.

In an appropriate coordinate patch we may introduce complex coordinates  $z^0, z^1, z^2$  and a Kähler potential  $K$  so that the initial Calabi-Yau metric is

$$(61) \qquad g = 4 \partial_a \bar{\partial}_b K \, dz^a d\bar{z}^b ,$$

where  $a, b \in \{0, 1, 2\}$ . We may also choose  $z^0 = x + i\tau$ , where  $\partial_\tau$  generates the  $U(1)$  isometry, and correspondingly  $\partial_\tau K = 0$ . In terms of our earlier notation, it is then straightforward to compute

$$(62) \qquad V = \frac{1}{\partial_x^2 K} , \qquad A = \frac{i}{\partial_x^2 K} (\bar{\partial}_i \partial_x K \, d\bar{z}^i - \partial_i \partial_x K \, dz^i) ,$$

where  $i = 1, 2$ . Notice here that, although both  $z^a$  and  $K$  are defined only locally in each coordinate patch, with Kähler transformations acting on  $K$  between patches, nevertheless the quantity  $\partial_x^2 K$  is a globally defined

function on  $X$ . This is clear, since it is the square length of  $\partial_\tau$ , which is a globally defined vector field by assumption. We also note that

$$(63) \quad e_\perp = d(\partial_x K) .$$

Since  $e_\perp$  is a global one-form on  $X$ , it follows that  $\partial_x K$  in different Kähler coordinate patches differ by an additive constant. Notice we may also write

$$(64) \quad V e_\perp = dx + \frac{1}{\partial_x^2 K} (\partial_i \partial_x K dz^i + \bar{\partial}_i \partial_x K d\bar{z}^i) .$$

We now turn to the rotated holomorphic  $(3,0)$ -form, which after rewriting (51) is

$$(65) \quad e^{-2\Phi'} \Omega' = [(V + \tan^2 \delta) e_\perp + i(dw + A)] \wedge V^{1/2} \Omega_\perp .$$

This then differs from  $\Omega$  in (49) only in the  $\tan^2 \delta$  term. Using the expression for  $A$  in (62) together with (64) this is

$$(66) \quad \begin{aligned} e^{-2\Phi'} \Omega' &= \left( dx + \tan^2 \delta e_\perp + idw + \frac{2}{\partial_x^2 K} \partial_i \partial_x K dz^i \right) \wedge V^{1/2} \Omega_\perp , \\ &= (dx + \tan^2 \delta e_\perp + idw) \wedge V^{1/2} \Omega_\perp . \end{aligned}$$

Here in the second line we have used that the closed complex two-form  $V^{1/2} \Omega_\perp$  is proportional to  $dz^1 \wedge dz^2$ . We thus see that  $e^{-2\Phi'} \Omega'$  is equal to the original  $\Omega$  provided we make the coordinate change

$$(67) \quad x' = x + \tan^2 \delta \partial_x K .$$

For, then

$$(68) \quad e^{-2\Phi'} \Omega' = (dx' + idw) \wedge V^{1/2} \Omega_\perp ,$$

which is the same as the formula for  $\Omega$  but with  $x$  replaced by  $x'$  and  $\tau$  replaced by  $w$ . Recall that the latter is already part of the diffeomorphism between  $X$  and  $X'$  discussed in the previous sections. Notice that in (67) the second term  $\partial_x K$  on the right hand side is a globally defined smooth function, up to constant shifts between coordinate patches. The latter are simply trivial constant shifts of  $x$ , and thus (67) defines a global diffeomorphism of  $X$  and  $X'$  which takes  $\Omega$  into  $e^{-2\Phi'} \Omega'$ .<sup>9</sup>

The coordinate transformation (67) takes a particularly interesting form if the original Calabi-Yau metric is *toric*, *i.e.*  $U(1)^3$  invariant. In this case there is a symplectic coordinate system in which

$$(69) \quad \omega = dy^a \wedge d\phi^a ,$$

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<sup>9</sup> An alternative argument is to note that if  $b_1(X) = 0$  then  $e_\perp = df$  for some *global* function  $f$ , and then  $x' = x + \tan^2 \delta f$ .

where without loss of generality we may take  $\phi^0 = \tau$ . The corresponding complex coordinates, with  $z^a = x^a + i\phi^a$ , are related to the symplectic coordinates via

$$(70) \quad y^a = \frac{\partial K}{\partial x^a} ,$$

where one can take  $K = K(x^a)$  to be  $U(1)^3$ -invariant. Thus (67) may be rewritten

$$(71) \quad \cos^2 \delta x' = \cos^2 \delta x + \sin^2 \delta y ,$$

where  $y = y^0$  is the symplectic coordinate paired with  $\tau = \phi^0$ . In this sense, the coordinate transformation mixes complex and symplectic coordinates.

In summary, the rotation in fact preserves the underlying complex manifold.<sup>10</sup> However, it certainly changes the Kähler structure to a non-Kähler structure. In particular, we note that

$$(72) \quad \exp(i\omega') = \exp \left[ -\frac{1}{(1 + \cot^2 \delta V)} e_{\perp} \wedge (dw + A) \right] \wedge \exp(i\omega) ,$$

as usual with  $\tau$  understood to be replaced by  $w$  in  $\omega$  on the right hand side. Thus the rotated pure spinor  $e^{i\omega'}$  is related to the original pure spinor  $e^{i\omega}$  via a simple multiplying form. Compare this with the rotation in [6], where the same pure spinor instead picks up a *phase* in the rotated solution.

### 4.3 Examples

In this section we present simple explicit examples of non-Kähler deformations of Calabi-Yau geometries. Namely, we discuss the Gibbons-Hawking and resolved conifold metrics.

#### The Gibbons-Hawking metric

The Gibbons-Hawking metric is a hyper-Kähler metric in four real dimensions with a tri-holomorphic Killing vector field, *i.e.* a Killing vector field  $\partial_{\tau}$  that preserves the triplet of complex structures implied by an integrable  $SU(2)$  structure in four dimensions. The metric takes the well-known form

$$(73) \quad g = \frac{1}{V} (d\tau + A)^2 + V g_{\mathbb{E}^3} ,$$

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<sup>10</sup>In fact this had to be the case for toric manifolds. Here it is a standard fact that there is a unique complex structure that is compatible with the torus action.

where  $V$  is any harmonic function on Euclidean three-space  $\mathbb{E}^3$ , and  $dA = - *_3 dV$ . By taking

$$(74) \quad V = \sum_{i=1}^m \frac{1}{|\mathbf{x} - \mathbf{x}_i|} ,$$

with  $\mathbf{x}_i \in \mathbb{E}^3$  distinct points, this is the family of asymptotically locally Euclidean metrics on the resolution of the  $A_{m-1}$  singularity  $\mathbb{C}^2/\mathbb{Z}_m$ .

Applying the rotation, we obtain the string frame solution

$$(75) \quad \begin{aligned} g' &= \frac{1}{Vh^2} (d\tau' + \cos^2 \delta A)^2 + V g_{\mathbb{E}^3} \\ &= \frac{1}{V'} (d\tau' + \cos^2 \delta A)^2 + V g_{\mathbb{E}^3} , \\ e^{-2\Phi'} &= h = \cos^2 \delta + \frac{\sin^2 \delta}{V} , \\ *H' &= -d \log h . \end{aligned}$$

These satisfy a four-dimensional version [10] of the equations (42), (43), (44), with  $\Omega$  now being the holomorphic  $(2,0)$ -form and equation (43) replaced by  $d(e^{-2\Phi'} \omega') = 0$ . In the obvious orthonormal frame with

$$(76) \quad e'_0 = \frac{1}{\sqrt{V}h} (d\tau' + \cos^2 \delta A) , \quad e'_i = \sqrt{V} dx_i , \quad i = 1, 2, 3 .$$

the gauge field curvature is given by

$$(77) \quad \mathcal{F}' = \frac{\sin \delta \cos \delta}{V^2 h} \left( \partial_i V e'_0 \wedge e'_i - \frac{1}{2!} \epsilon^{ijk} \partial_k V e'_i \wedge e'_j \right) .$$

From this expression it is straightforward to check that  $\mathcal{F}'$  is both type  $(1,1)$  and satisfies the HYM equation  $\omega' \lrcorner \mathcal{F}' = 0$ . One also checks that the Bianchi identity  $dH' = \mathcal{F}' \wedge \mathcal{F}'$  holds. In fact, more precisely this holds for all  $\delta \in [0, \pi/2)$ . For the T-duality limit with  $\delta = \pi/2$  we have  $\mathcal{F}' = 0$ , but the Bianchi identity for the solution with harmonic function (74) then has  $m$  delta-function sources on the right hand side; *cf.* the comments in section 3.4. This is as expected, since the T-dual solution is  $m$  five-branes in flat spacetime, smeared over a circle  $S^1$ , and positioned at the points  $\mathbf{x}_i \in \mathbb{R}^3$ , where  $\mathbb{R}^3$  is transverse to all the five-branes and the  $S^1$  over which the branes are smeared.

Any such solution in four dimensions is necessarily *conformally* hyper-Kähler [10]. This is clear from the second form of the metric in (75), since the rotated  $Vh = \cos^2 \delta V + \sin^2 \delta$  is also harmonic, and thus of Gibbons-Hawking type.

### The resolved conifold

Next, as an explicit example in dimension six, we consider the rotation of the Ricci-flat metric on the resolved conifold [43]. A similar analysis can be done for the Ricci-flat Kähler metrics presented in [44]. The resolved conifold metric can be written in the following explicit form

$$\begin{aligned} ds_{\text{RC}}^2 &= \frac{1}{\kappa(r)} dr^2 + \frac{r^2}{6} (d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + \left( \alpha^2 + \frac{r^2}{6} \right) (d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) \\ (78) \quad &+ \frac{r^2}{9} \kappa(r) (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2, \end{aligned}$$

where

$$(79) \quad \kappa(r) = \frac{9\alpha^2 + r^2}{6\alpha^2 + r^2}.$$

The (asymptotically Reeb) Killing vector  $\partial/\partial\psi$  does not leave the holomorphic  $(3,0)$ -form  $\Omega$  invariant, and hence we can apply our rotation only to an arbitrary linear combination of  $\partial/\partial\phi_1$  and  $\partial/\partial\phi_2$ . We then first change coordinates, making a transformation

$$\begin{aligned} \phi_1 &= a\varphi_1 + b\varphi_2, \\ (80) \quad \phi_2 &= c\varphi_1 + d\varphi_2. \end{aligned}$$

We will momentarily perform the rotation along  $\partial/\partial\varphi_1$ , without loss of generality, which recall is required to generate a  $U(1)$  action. As such, the transformation (80) should be taken to lie in  $SL(2, \mathbb{Z})$ , although for the local computations that follow this comment is not important. We next write the metric adapted to the Killing vector  $\partial/\partial\varphi_1$  in the form (48), and compute

$$\begin{aligned} V^{-1} &= a^2 \frac{r^2}{6} \sin^2 \theta_1 + c^2 \left( \alpha^2 + \frac{r^2}{6} \right) \sin^2 \theta_2 \\ (81) \quad &+ \frac{r^2}{9} \kappa(r) (a \cos \theta_1 + c \cos \theta_2)^2. \end{aligned}$$

Notice that provided  $c \neq 0$ , this vanishes at the north ( $\theta_2 = \pi$ ) and south ( $\theta_2 = 0$ ) poles of the resolved two-sphere. Of course, the resolved conifold metric is nevertheless perfectly smooth at these points. We also have

$$\begin{aligned} A &= V \left[ \left( ab \frac{r^2}{6} \sin^2 \theta_1 + cd \left( \alpha^2 + \frac{r^2}{6} \right) \sin^2 \theta_2 + \right. \right. \\ &\quad \left. \frac{r^2}{9} \kappa(r) (a \cos \theta_1 + c \cos \theta_2) (b \cos \theta_1 + d \cos \theta_2) \right) d\varphi_2 + \\ (82) \quad &\left. \frac{r^2}{9} \kappa(r) (a \cos \theta_1 + c \cos \theta_2) d\psi \right]. \end{aligned}$$



Let us now compute the volume of the resolved two-sphere as a function of the deformation parameter  $\delta$ . The induced metric at  $r = 0$  reads

$$(83) \quad ds^2|_{S^2} = \alpha^2(d\theta_2^2 + f(\theta_2)\sin^2\theta_2 d\phi_2^2),$$

where

$$(84) \quad f(\theta_2) = \frac{1}{(1 + c^2\alpha^2 \tan^2 \delta \sin^2 \theta_2)^2}.$$

The volume of the two-sphere may be computed analytically and is given by

$$(85) \quad \text{vol}(S_\delta^2) = (4\pi\alpha^2) \frac{\text{arctanh}\left[\frac{c\alpha \tan \delta}{\sqrt{1+c^2\alpha^2 \tan^2 \delta}}\right]}{c\alpha \tan \delta \sqrt{1+c^2\alpha^2 \tan^2 \delta}}.$$

This is the volume of the two-sphere of the resolved conifold metric, times a function that is monotonically decreasing between one at  $\delta = 0$  and zero at  $\delta = \pi/2$ . The non-Kähler deformation therefore squashes the two-sphere, and in the limit  $\delta = \pi/2$  the volume goes to zero.

This is an appropriate point to comment in more detail on the limit  $\delta \rightarrow \pi/2$ . In section 3.4, and then throughout the paper, we have chosen to identify coordinates in the rotated solution in such a way that for  $\delta \in [0, \pi/2)$  the underlying manifold remains fixed. Indeed, we showed moreover in section 4.2 that the complex structure is then also invariant. Generically, this will be the only way to obtain a *regular* supergravity solution for general  $\delta$ .<sup>11</sup> However, with these global identifications the limit  $\delta \rightarrow \pi/2$  is *singular*, and not in fact the T-dual solution. The reason for this is simple: in general the T-dual solution requires a *different* identification of coordinates, in order to obtain a regular supergravity solution. This is most clear when the T-dual solution has different topology; manifestly, one cannot then obtain the T-dual solution as a smooth limit  $\delta \rightarrow \pi/2$  of smooth supergravity solutions, as the underlying manifolds have different topology. More computationally, the metric in the  $w$  coordinates (35) is clearly degenerate at  $\delta = \pi/2$ . Instead the regular T-dual metric in this limit is described by giving a finite periodicity to the original  $\tau' = w \cos^2 \delta$  coordinate.

Returning to the conifold example, the most interesting case is perhaps that with  $a = c = d = 1$ ,  $b = 0$ . In this case, and as is well-known, the T-dual solution describes the back-reaction of two five-branes in flat spacetime. In the above coordinates, the two five-branes arise from the codimension four fixed point sets at  $\{\theta_1 = 0, \theta_2 = \pi\}$ ,  $\{\theta_1 = \pi, \theta_2 = 0\}$ , which are two copies

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<sup>11</sup>A general analysis here splits into different cases. However, the main issue is the loci  $S$  where  $\partial_\tau$  has fixed points. If one tries to take  $w$  to have any period other than that of  $\tau$ , then one will obtain conical deficit singularities along  $S$ , leading to a singular supergravity solution.

of  $\mathbb{R}^2$  parametrized by  $r, \psi$ . These become the five-brane worldvolumes in the T-dual solution, while the minimal  $S^2$  maps to an arc that joins the five-branes.

Let us briefly comment also on the  $B$ -field and gauge field. Notice that the pull-back of the  $B$ -field to the two-sphere vanishes. On the other hand, the pull-back of the gauge field is non-zero and reads

$$(86) \quad \mathcal{F}'|_{S^2} = c\alpha^2 \tan \delta \, d \left[ \frac{\sin^2 \theta_2}{1 + c^2 \alpha^2 \tan^2 \delta \sin^2 \theta_2} d\phi_2 \right] .$$

However, the integral

$$(87) \quad \int_{S^2} \mathcal{F}' = 0 .$$

Notice that if this integral were non-zero then the parameter  $\delta$  would have been quantized. That it is not is in agreement with our general discussion of global properties of the deformed Calabi-Yau metrics in section 3.4.

## 5 Rotating heterotic string solutions: including $\alpha'$

In this section we restore the  $\alpha'$  of section 2, and carefully describe in what sense the rotated solutions in the previous two sections are solutions to the low-energy limit of heterotic string theory.

### 5.1 $\alpha'$ expansions

The supergravity fields in section 2 of course have an expansion in  $\alpha'$ . These are Taylor expansions around  $\alpha' = 0$ , and in general we denote the coefficient of  $\alpha'^n$  with a subscript  $n$ . Thus, for example, the metric is  $g = g_0 + \alpha' g_1 + O(\alpha'^2)$ . In particular, we note that the Bianchi identity (5) implies that

$$(88) \quad dH_0 = 0 .$$

A computation from the definitions shows that

$$(89) \quad R_{ijkl}^+ - R_{klij}^- = \frac{1}{2}(dH)_{ijkl} .$$

Notice the index structure. Thus to *zeroth order*  $R_{0ijkl}^+ = R_{0klij}^-$ . The integrability condition for the gravitino equation immediately implies that

$$(90) \quad R_{0ijkl}^+ \Gamma^{kl} \epsilon = 0 ,$$

which is the familiar statement of holonomy reduction for the Bismut connection  $\nabla^+$ , and thus from (89) we have that

$$(91) \quad R_{0klij}^- \Gamma^{kl} \epsilon = 0 .$$

In other words, viewing  $\mathcal{R}^-$  as the curvature of a connection on the tangent bundle, then this connection formally satisfies the last *instanton equation* in (6) to zeroth order, with  $\mathcal{F}$  replaced by  $\mathcal{R}_0^-$ .

Also notice that in the Bianchi identity (5), and similarly the equations of motion (3), one can effectively replace  $\mathcal{R}^-$  by  $\mathcal{R}_0^-$ . This is simply because the  $\alpha'$  corrections to the latter are at order  $\alpha'^2$  in both (3), (5), and may hence be absorbed into  $O(\alpha'^2)$ . This has the important consequence that the connection that appears in these higher derivative terms satisfies the instanton equation. In appendix A we show directly that if the supersymmetry equations and Bianchi identity are satisfied, up to and including first order in  $\alpha'$  and with Bianchi identity

$$(92) \quad dH = 2\alpha' (\text{tr } \mathcal{F} \wedge \mathcal{F} - \text{tr } \mathcal{R}_0^- \wedge \mathcal{R}_0^-) + O(\alpha'^2) ,$$

then the equations of motion (3) follow. For earlier results, see [37, 38].

The above comments lead to the following simple consequence. Suppose one has an exact supersymmetric solution to the above equations with  $\mathcal{A} = 0$  and  $\alpha' = 0$ . In other words, we have a supersymmetric type II solution, with  $dH = 0$ . We may now consider turning on a non-Abelian gauge field by setting  $\mathcal{F} = \mathcal{R}^- \equiv \mathcal{R}_0^-$  (at this point we have set  $\alpha' = 0$ ), which by virtue of the above discussion satisfies the instanton equation  $\mathcal{F}_{ij}\Gamma^{ij}\epsilon = 0$ . Also, since  $\mathcal{F} = \mathcal{R}^-$ , the  $O(\alpha')$  part of the Bianchi identity (5) is identically zero. Reinstating  $\alpha' \neq 0$ , we thus automatically obtain a supersymmetric solution to the heterotic string up to and including  $O(\alpha')$ . Moreover, this solution solves *exactly* the above heterotic equations, with all  $O(\alpha'^2)$  terms set to zero.<sup>12</sup> This is a slightly modified form of the usual standard embedding. Notice that it is crucial that it is Hull's connection that appears in the higher derivative Bianchi identity, rather than say the Bismut or Levi-Civita connection. In this case the non-Abelian gauge field takes values in a  $Spin(6) \cong SU(4) \subset E_8$  subgroup.

We conclude this subsection with an aside comment, which we feel is nevertheless important to make. In the literature connections other than  $\mathcal{R}^-$  have been taken in the Bianchi identity (5). In particular, the *Chern connection* has been used. This requires some further comment. Suppose that the spacetime takes the product form  $\mathbb{R}^{1,3} \times X$ . Then, as reviewed in section 4,  $X$  is a complex manifold equipped with in general a non-Kähler structure. The supersymmetry equations imply in particular that  $dH$  has Hodge type (2,2) with respect to the integrable complex structure. Thus in order to solve the Bianchi identity (5), up to and including order  $\alpha'$ , it follows that the curvature tensor that appears in this Bianchi identity should

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<sup>12</sup>Of course, these terms are not zero in heterotic string theory.

also have Hodge type (2,2). This is indeed true of the Chern connection. However, the Chern connection is *not* in general an instanton. In fact in the references in [39] the following identity is proven:

$$(93) -\frac{1}{2}R^C_{ijkl}\omega^{ij}\omega^{kl} = -\frac{1}{2}R^+_{ijkl}\omega^{ij}\omega^{kl} + C_{ijk}C^{ijk} + \frac{1}{4}(\mathrm{d}H)_{ijkl}\omega^{ij}\omega^{kl}.$$

Here  $\omega$  is the type (1,1) (non)-Kähler form,  $R^C$  denotes the curvature of the Chern connection, and  $C$  denotes its torsion. If the gravitino equation and Bianchi identity hold, up to order  $\alpha'$ , *and* the Chern connection is an instanton (at zeroth order), then the zeroth order part of (93) implies that  $C_0 = 0$ . But

$$(94) \quad C_{ijk} = \frac{1}{2} (J^m_i(\mathrm{d}\omega)_{mjk} + J^m_j(\mathrm{d}\omega)_{imk}) ,$$

where  $J$  denotes the complex structure tensor. Thus if  $C_0 = 0$  then at zeroth order the solution is necessarily Kähler. In general this is not true! The problem with choosing the Chern connection in the Bianchi identity is then that supersymmetry and the Bianchi identity do *not* imply the correct equations of motion – again, we refer also to appendix A.

## 5.2 Rotations to first order in $\alpha'$

Comparing the heterotic string equations of motion (3) with the Abelian heterotic supergravity equations (14), one sees that they are essentially the same, up to some small, but important, differences. Firstly, the heterotic string gauge field  $\mathcal{A}$  is non-Abelian, while in the Abelian heterotic supergravity theory that embeds into type II, the gauge field  $\mathcal{A}$  is Abelian. Of course, these are easily related by simply restricting  $\mathcal{A}$  to lie in a  $U(1) \cong SO(2)$  subgroup of the heterotic gauge group. Secondly, the gauge field in the heterotic string is accompanied by factors of  $\alpha'$  in the action. Thirdly, the heterotic string also has higher derivative terms at this order in  $\alpha'$ .

Let us now turn to our rotated Abelian heterotic solutions, focusing on the simpler case (33) in section 3.3. In order not to cause confusion in what follows, we keep the primes in (33), now regarding these primes as denoting an Abelian heterotic supergravity solution, without any  $\alpha'$ .

We note first that when  $\alpha' = 0$ , the gauge field no longer appears in the low-energy action of the heterotic string. This suggests that in order to introduce an  $\alpha'$  dependence into the rotated solutions (33), we set the rotation angle

$$(95) \quad \delta = \sqrt{\alpha'} \lambda .$$

This is simply because when  $\alpha' = 0$  the solution should have zero gauge field, which implies that  $\delta$  should also be zero. We then define

$$\begin{aligned} \mathcal{A} &= \frac{1}{\sqrt{\alpha'}} \mathcal{A}' , \\ (96) \quad &= -\lambda \left[ d\tau' - \frac{1}{V} (d\tau' + A) \right] + O(\alpha') . \end{aligned}$$

Here one should be more precise about how the  $U(1) \cong SO(2)$  gauge field  $\mathcal{A}'$  is embedded into the heterotic gauge group. Obviously, there are choices. For example, we may more precisely write

$$(97) \quad \mathcal{A} = -\lambda \left[ d\tau' - \frac{1}{V} (d\tau' + A) \right] \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & & \\ \frac{i}{\sqrt{2}} & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} + O(\alpha') .$$

The purely imaginary, skew-symmetric matrix here embeds the  $U(1)$  gauge field  $\mathcal{A}'$  as an  $SO(2)$  subgroup of either  $SO(32)$  or  $SO(16) \subset E_8$ . However, most of what we say will be independent of these details, and we thus suppress this for the time being. The explicit introduction of  $\sqrt{\alpha'}$  in (96) and normalization in (97) ensures that the kinetic terms are then related as  $\mathcal{F}'_{ij} \mathcal{F}'^{ij} = \alpha' \text{tr } \mathcal{F}_{ij} \mathcal{F}^{ij}$ , and similarly now the Bianchi identity  $dH' = \mathcal{F}' \wedge \mathcal{F}'$  becomes  $dH' = \alpha' \text{tr } \mathcal{F} \wedge \mathcal{F}$ . Since the other fields already appear at order zero, we simply set  $g = g'$ ,  $H = H'$  and  $\Phi = \Phi'$ , where the primed fields now depend on  $\alpha'$  via (95). The full solution is then easily computed to be

$$\begin{aligned} g &= g_{\text{CY}} - \frac{2\alpha'\lambda^2}{V} (d\tau' + A) (d\tau' + A - \frac{V}{2} d\tau') + O(\alpha'^2) , \\ e^{2\Phi} &= e^{2\Phi_0} \left[ 1 - \frac{\alpha'\lambda^2}{V} (1 - \frac{V}{2}) \right] + O(\alpha'^2) , \\ \mathcal{A} &= -\lambda \left[ d\tau' - \frac{1}{V} (d\tau' + A) \right] + O(\alpha') , \\ (98) \quad B &= -\frac{\alpha'\lambda^2}{V} d\tau' \wedge A + O(\alpha'^2) . \end{aligned}$$

This is a one-parameter family of deformations of the original Calabi-Yau metric, with deformation parameter  $\lambda$ . At zeroth order the solution is Calabi-Yau, with  $B_0 = 0$ . Notice that the zeroth order gauge field  $\mathcal{A}_0$  is non-zero; however, at this order the gauge field does not appear in the action. By construction, (98), *including* the infinite  $\alpha'$  expansion coming from the original solution (33), solves the supersymmetry equations (6) *exactly* up to and including first order in  $\alpha'$ . However, from a physical perspective one should truncate the solution at order  $\alpha'^2$ , as we have done in (98), since the supersymmetry equations will receive corrections at this order.

However, the Bianchi identity (5) is solved *without* the higher derivative terms in  $\mathcal{R}^-$ . We may try to incorporate this using a modification of the standard embedding. Notice that  $\mathcal{R}_0^- = \mathcal{R}_{\text{CY}}$  is simply the curvature tensor of the original Calabi-Yau metric, and as such indeed solves the instanton equation at *zeroth order*. This is also an  $SU(3)$  connection. Perhaps the minimal choice of “modified standard embedding” is then to write the following  $SU(4) \subset E_8$  gauge field

$$(99) \quad \mathcal{A}_{\text{MSE}} = \frac{1}{\sqrt{\alpha'}} \mathcal{A}' \mathcal{Q}_4 + \omega_{\text{CY}}^{\text{spin}} ,$$

where

$$(100) \quad \mathcal{Q}_4 = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & 0 & 0 & & \\ 0 & -\frac{1}{2\sqrt{3}} & 0 & 0 & & \\ 0 & 0 & -\frac{1}{2\sqrt{3}} & 0 & & \\ 0 & 0 & 0 & -\frac{1}{2\sqrt{3}} & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} .$$

Here the spin connection of the Calabi-Yau manifold  $\omega_{\text{CY}}^{\text{spin}}$ , which is an  $SU(3)$  connection, is understood to be embedded in the  $3 \times 3$  block that commutes with  $\mathcal{Q}_4$ . Then we have that

$$(101) \quad \mathcal{F}_{\text{MSE}} = \mathcal{R}_{\text{CY}} + \frac{1}{\sqrt{\alpha'}} \mathcal{Q}_4 \mathcal{F}' ,$$

and the normalizations guarantee that

$$(102) \quad \begin{aligned} dH &= \mathcal{F}' \wedge \mathcal{F}' \\ &= \alpha' (\text{tr } \mathcal{F}_{\text{MSE}} \wedge \mathcal{F}_{\text{MSE}} - \text{tr } \mathcal{R}^- \wedge \mathcal{R}^-) + O(\alpha'^2) , \end{aligned}$$

and thus we solve the Bianchi identity at the appropriate order. Notice here that it is crucial that the  $U(1)$  gauge field commutes with the  $SU(3)$  spin connection of the original Calabi-Yau metric in (99), ensuring that these terms do not mix in the wedge product and associated matrix multiplication. Of course, it is also important that both are traceless.

The first term in (99) solves the instanton equation up to and including  $O(\alpha')$ , by construction. However,  $\mathcal{R}_{\text{CY}}$  is only an instanton *at zeroth order*. Thus at this point we have solved all equations up to and including  $O(\alpha')$ , *except* the instanton equation, which is solved only at zeroth order. However, notice that the gauge field  $\mathcal{A}$  enters the action (2) already at order  $\alpha'$ . Thus an  $O(\alpha')$  correction to  $\mathcal{A}$  enters the action at *second order*, while the corresponding  $O(\alpha')$  corrections to the metric, dilaton and  $B$ -field enter

at first order. In this sense, we already have a solution up to and including  $O(\alpha')$ . Nevertheless, we shall briefly comment further on higher order corrections below.

The gauge field (99) lies in a  $U(1) \times SU(3) \subset SU(4) \subset E_8$  subgroup of the full heterotic gauge group. The maximal commuting subgroup in  $E_8$  is then  $SO(10) \times U(1)$ , where  $SO(10)$  is the commutant of  $SU(4)$  in  $E_8$  and the  $U(1)$  factor is the same as that in (99). Indeed, an alternative way to think about this is that the initial solution with  $\delta = 0$  has only an  $SU(3)$  gauge field turned on, via the standard embedding, and as such has commutant  $E_6$ . Then the rotation turns on an additional  $U(1)$  gauge field which breaks this to  $SO(10) \times U(1)$ . Although our solutions here are non-compact, one might imagine that these are good *local models* for a compact non-Kähler heterotic solution. For example, the conifold metric is believed to model a neighbourhood of a compact Calabi-Yau manifold near to a conifold transition, and our rotation then applies directly to the resolved and deformed conifolds. In this setting, and assuming our gauge field extends globally as a  $U(1) \times SU(3)$  or  $SU(4)$  gauge field, then the low-energy gauge group in  $\mathbb{R}^{1,3}$  is  $SO(10)$ . In particular, in the former case the  $U(1)$  commutant obtains a mass via the Green-Schwarz mechanism – at least, this is so in the Calabi-Yau case, as discussed, for example, in [45]. Thus the rotation effectively breaks the initial  $E_6$  gauge group to  $SO(10)$ .

A slight generalization of the above construction can be given when the initial solution has non-zero  $H_0$ . As explained in section 5.1, via the standard embedding using Hull’s connection in this case the initial non-Abelian gauge field lies in an  $SU(4)$  subgroup (the holonomy of the  $\omega_0^-$  connection) of  $E_8$ . The rotation turns on an Abelian gauge field via the embedding

$$(103) \quad \mathcal{A}_{\text{MSE}} = \frac{1}{\sqrt{\alpha'}} \mathcal{A}' \mathcal{Q}_5 + \omega_0^- ,$$

where  $\mathcal{Q}_5 = \frac{1}{\sqrt{20}} \text{diag}(4, -1, -1, -1, -1, 0, \dots)$ , thus breaking the low-energy gauge group to  $SU(5)$ . In addition to [9], solutions amenable to this construction include the  $T^2$  bundles over conformally hyper-Kähler two-folds, presented explicitly in [46, 10].

We conclude this section with a brief comment on the possibility of correcting (99) at order  $\alpha'$  to solve the instanton equation at this order. Notice that this correction, whatever it is, does *not* affect the Bianchi identity (102), since such a correction enters at  $O(\alpha'^2)$  in the Bianchi identity. The

philosophy from here follows the important, but relatively unknown, paper by Witten and Witten [14].<sup>13</sup>

We write the non-Abelian gauge field, up to and including order  $\alpha'$ , as

$$(104) \quad \mathcal{A}^{(1)} = \mathcal{A}_{\text{MSE}} + \alpha' S ,$$

where  $S$  is an  $\alpha'$ -independent one-form with values in the Lie algebra of  $SU(4)$ . Notice here that  $\mathcal{A}_{\text{MSE}}$ , as we have defined it, contains all powers of  $\alpha'$ . In particular, the (perhaps confusing) notation above implies that  $\mathcal{A}_0^{(1)} = \mathcal{A}_{\text{MSE}0}$  and  $\mathcal{A}_1^{(1)} = \mathcal{A}_{\text{MSE}1} + S$ . Expanding everything in powers of  $\alpha'$  and collecting all linear terms we obtain the equations

$$(105) \quad \begin{aligned} \Omega_0 \wedge \mathcal{F}_1^{(1)} &= -\Omega_1 \wedge \mathcal{F}_0^{(1)} , \\ \omega_0 \wedge \omega_0 \wedge \mathcal{F}_1^{(1)} &= -2\omega_0 \wedge \omega_1 \wedge \mathcal{F}_0^{(1)} . \end{aligned}$$

The left hand sides of these equations are just the usual instanton equations, and vanish if one replaces  $\mathcal{F}_1^{(1)}$  by  $\mathcal{F}_0^{(1)}$ , by construction. The right hand sides are then “source” terms. One could decompose  $\mathcal{F}_1^{(1)}$  into irreducible representations with respect to the zeroth order structure, and use the equations above to determine its components. However,  $\mathcal{F}_1^{(1)}$  is not arbitrary, since the Bianchi identity for the gauge field implies

$$(106) \quad \mathcal{F}_1^{(1)} = d\mathcal{A}_1^{(1)} + \mathcal{A}_{\text{MSE}0} \wedge \mathcal{A}_1^{(1)} + \mathcal{A}_1^{(1)} \wedge \mathcal{A}_{\text{MSE}0} .$$

Notice that the equations (105) are linear, first order PDEs for the connection  $S$ , where all other terms are known from the rotated solution. It would be interesting to investigate this system further.

## 6 Discussion

In this paper we have studied a solution-generating transformation in the context of heterotic supergravity, with a non-trivial  $U(1)$  gauge field in the Cartan subgroup of  $E_8 \times E_8$  or  $SO(32)$ . In particular, we have discussed how this transformation preserves supersymmetry. In the case of backgrounds of the form  $\mathbb{R}^{1,3} \times X$  we have explicitly shown that solutions obtained as rotations of Calabi-Yau geometries satisfy the non-Kähler equations, and that the complex structure remains invariant.

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<sup>13</sup>Reference [14] is crucial for the consistency of Calabi-Yau compactifications in which one takes a Hermitian-Yang-Mills gauge field that is *not* the standard embedding. Such a solution leads to  $dH_1 \neq 0$ , which means that the metric is Calabi-Yau only to leading order. Despite the large number of papers on this subject, [14] only has 21 references on SPIRES at the time of writing.



One of the main observations we have made is that heterotic solutions with an Abelian gauge field can be formally mapped to type II solutions, where the heterotic gauge field becomes a component of the metric in an internal space of one dimension higher (see also [10]). Based on this, we have seen that an  $O(2, 1)$  subgroup of the (Abelian) heterotic duality group  $O(d + 16, d)$  is effectively embedded into an  $O(2, 2)$  subgroup of the type II T-duality group. We thus realize the heterotic transformation as a simple combination of rotations and ordinary Buscher T-duality in type II theories. This relationship perhaps deserves to be further studied. For example, it would be interesting to investigate whether a suitable notion of heterotic generalized geometry exists based on the  $O(d + 16, d)$  duality group, generalizing the generalized geometry that puts the metric and  $B$ -field on the same footing in type II.

In the context of heterotic string theory, taking into proper account the parameter  $\alpha'$  leads to some modifications of the discussion. Firstly, although the solution-generating formulae formally contain an infinite series<sup>14</sup> in powers of  $\alpha'$ , the equations of motion, supersymmetry variations and Bianchi identities for the heterotic theory are only known up to some low order in the  $\alpha'$  expansion [35]. Therefore, the transformation only makes sense if the expansions are truncated at the appropriate order. In addition, the duality transformations do not apply to non-Abelian gauge fields. As we discussed, it turns out that it is quite simple to remedy this, and obtain supersymmetric heterotic solutions at *first order* in  $\alpha'$ , including a non-Abelian gauge field with a slightly modified standard embedding. Thus, given any (non-compact) Calabi-Yau geometry, we have constructed non-Kähler solutions, breaking the heterotic gauge group from  $E_6$  (for the Calabi-Yau with standard embedding) to the GUT gauge group  $SO(10)$ .

Going back to the type II setting, the transformation we discussed is then *exact*, since the gauge field is just a metric component and hence appears at the same (lowest) order in  $\alpha'$  here. We may then start in type II with the direct product of a Calabi-Yau geometry with a circle, and the transformation will “twist” this circle over the base Calabi-Yau (although topologically this twisting is always trivial), with the base becoming precisely a non-Kähler geometry. Furthermore, we can start with any supersymmetric solution with non-trivial  $B$ -field, and also perform the transformation.<sup>15</sup> For example, one could apply this rotation to the Maldacena-Nuñez solution [9].

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<sup>14</sup>In [29] it is suggested that the symmetry should be valid at all orders in  $\alpha'$ , although the precise transformation of the fields may be corrected.

<sup>15</sup>In this case the twisting could change the global topology of the transformed solution.

Notice that if we start with a type IIB solution of the form  $\mathbb{R}^{1,2} \times S_z^1 \times X$ , a T-duality along  $S_z^1$  will give a  $\mathbb{R}^{1,2} \times \tilde{S}_z^1 \times X$  geometry in type IIA. We can then perform a rotation in the  $(z, \tau)$  plane, and finally another T-duality along the rotated  $\tilde{S}_z^1$ . Let us call this a TrT transformation. If we perform a further rotation of the final type IIB solution, we have simply composed a T-duality with our solution-generating transformation. However, from a *physical* point of view, the second rotation does nothing, and hence effectively we have performed a TrT transformation. This is (locally) equivalent to a TsT transformation, as shown in appendix B. The latter can be interpreted in the dual field theory (when this exists) as a *dipole deformation* [21, 47]. In these cases one could reinterpret our results in the dual field theory. It would be interesting to explore applications of our results to five-brane solutions representing gravity duals of field theories.

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### Appendix A An integrability result

In this appendix we prove an integrability result in heterotic supergravity, extending the result in the appendix of [1] to include the Bianchi identity corrected at first order in  $\alpha'$ . This is a simplified version of the proof given in [38]. The details of the proofs of the latter reference differ depending on the dimension of the internal manifold, namely on the form of the ten-dimensional spacetime, which is taken to be of the product form  $\mathbb{R}^{1,p} \times X_{9-p}$ . In contrast, our “spinorial” proof is valid in any dimension  $p$ .

Let us record here the equations of motion at first order in  $\alpha'$

$$(107) \quad R_{ij} + 2\nabla_i \nabla_j \Phi - \frac{1}{4} H_{ikl} H_j^{kl} - 2\alpha' \text{tr } \mathcal{F}_{ik} \mathcal{F}_j^k + 2\alpha' \text{tr } \mathcal{R}_{ik} \mathcal{R}_j^k = 0 ,$$

$$(108) \quad \nabla^2(e^{-2\Phi}) - \frac{1}{6} e^{-2\Phi} H_{ijk} H^{ijk} - \alpha' e^{-2\Phi} \text{tr } \mathcal{F}_{ij} \mathcal{F}^{ij} + \alpha' e^{-2\Phi} \text{tr } \mathcal{R}_{ij} \mathcal{R}^{ij} = 0 ,$$

$$(109) \quad \nabla^i (e^{-2\Phi} H_{ijk}) = 0 ,$$

$$(110) \quad \nabla^{+i} (e^{-2\Phi} \mathcal{F}_{ij}) = 0 ,$$

and the modified Bianchi identity

$$(111) \quad dH = 2\alpha' (\text{tr } \mathcal{F} \wedge \mathcal{F} - \text{tr } \mathcal{R} \wedge \mathcal{R}) .$$

In this appendix we do not specify the connection  $\omega$  used to compute the curvature  $\mathcal{R} = d\omega + \omega \wedge \omega$ . We will see that the integrability results are compatible with  $\mathcal{R} = \mathcal{R}_0^-$ , namely the zeroth order part of the Hull curvature.

We shall not explicitly write the gauge group indices on the field strength  $\mathcal{F}_{ij}{}^{ab}$ . Similarly, we denote by  $\mathcal{R}_{ij}$  the curvature two-form

$$(112) \quad \mathcal{R}_{ij}{}^{ab} = R_{ijkl} e^{ak} e^{bl} ,$$

where  $R_{ijkl}$  is the Riemann tensor computed with the given connection. We record also the supersymmetry equations in these conventions:

$$(113) \quad \left( \nabla_i + \frac{1}{8} H_{ijk} \Gamma^{jk} \right) \epsilon = 0 ,$$

$$(114) \quad \left( \Gamma^i \partial_i \Phi + \frac{1}{12} H_{ijk} \Gamma^{ijk} \right) \epsilon = 0 ,$$

$$(115) \quad \mathcal{F}_{ij} \Gamma^{ij} \epsilon = 0 .$$

Notice we are using exactly the same conventions as [1]. We can then use equation (8.7) of this reference, which we record here:

$$(116) \quad \left( \nabla^2 \Phi - 2(\nabla \Phi)^2 + \frac{1}{12} H_{ijk} H^{ijk} + \frac{\alpha'}{2} \text{tr } \mathcal{F}_{ij} \mathcal{F}^{ij} \right) \epsilon = \\ - \frac{1}{48} (dH - 2\alpha' \text{tr } \mathcal{F} \wedge \mathcal{F})_{ijkl} \Gamma^{ijkl} \epsilon - \frac{1}{4} e^{2\Phi} \nabla^i (e^{-2\Phi} H_{ijk}) \Gamma^{jk} \epsilon .$$

This is derived using only the dilatino (114) and gaugino (115) supersymmetry equations. Let us now assume that the spacetime is of the form  $\mathbb{R}^{1,p} \times X_{9-p}$ . Then one of the equations following from supersymmetry is the calibration condition [10]

$$(117) \quad e^{-2\Phi} *_9 H = -d(e^{-2\Phi} \Xi) .$$

Here  $\Xi$  is a  $G$ -invariant form specifying, at least in part, the related  $G$ -structure. For example, when  $p = 3$  then  $\Xi = \omega$  is the type (1,1) form for the associated  $G = SU(3)$  structure, while when  $p = 2$  then  $\Xi = \phi$  is the associative three-form for the associated  $G = G_2$  structure; a complete discussion may be found in [10]. Equation (117) then automatically implies the  $H$  equation of motion. Hence the last term in (116) is zero. Now, if we assume that the Bianchi identity (111) and the dilaton equation of motion (108) hold, substituting them into (116) we obtain

$$(118) \quad \frac{\alpha'}{2} \text{tr } \mathcal{R}_{ij} \mathcal{R}^{ij} \epsilon = \frac{\alpha'}{24} (\text{tr } \mathcal{R} \wedge \mathcal{R})_{ijkl} \Gamma^{ijkl} \epsilon .$$

Using the identity

$$(119) \quad \{\Gamma_{ij}, \Gamma^{kl}\} = 2\Gamma_{ij}{}^{kl} - 4\delta_{ij}^{kl} ,$$

the latter equation can be written as

$$(120) \quad \{\Gamma^{ij}, \Gamma^{kl}\} \text{tr}(\mathcal{R}_{ij} \mathcal{R}_{kl}) \epsilon = 0 .$$

Multiplying on the left by  $\epsilon^\dagger$ , defining  $P = \Gamma^{ij} \mathcal{R}_{ij}$  and noting<sup>16</sup> that  $P^\dagger = -P$ , we obtain

$$(121) \quad \text{tr}[(P\epsilon)^\dagger P\epsilon] = 0 .$$

However, since the trace is positive definite this implies that  $P\epsilon = 0$ , which is the instanton equation

$$(122) \quad \mathcal{R}_{ij} \Gamma^{ij} \epsilon = 0 .$$

We conclude, using also the remaining results in the appendix of reference [1], that the supersymmetry variations (113), (114), (115), together with the Bianchi identity (111), imply the equations of motion (107) if and only if the curvature of the connection used in (111) and (107) is an *instanton*. In particular, to this order in the  $\alpha'$  expansion, Hull's curvature  $\mathcal{R}^-$  is singled out by compatibility of supersymmetry and the equations of motion.

## Appendix B TrT = TsT

In this appendix we show that, at least locally, a TrT transformation is equivalent to a TsT transformation, for any configuration without RR fields. Presumably this result extends to configurations with RR fields, although we have not examined this. The following computation is a straightforward, but slightly tedious, application of the T-duality rules.

We begin with a solution

$$(123) \quad \begin{aligned} ds^2 &= f_1(dx_1 + a dx_2 + A_1)^2 + f_2(dx_2 + A_2)^2 + ds_\perp^2 , \\ B &= B_1 \wedge dx_1 + B_2 \wedge dx_2 + b dx_1 \wedge dx_2 + B_\perp , \end{aligned}$$

where the metric is in string frame. We also have a non-trivial dilaton field  $\Phi$ .  $\partial/\partial x_1$  and  $\partial/\partial x_2$  are Killing vectors. The metric is written in a form adapted to performing a T-duality along the  $\partial/\partial x_1$  direction. We then have that  $A_1$ ,  $A_2$  and  $B_1$ ,  $B_2$  have zero contractions with  $\partial/\partial x_1$  and  $\partial/\partial x_2$ , and moreover these one-forms, together with the functions  $f_1, f_2, a, b$ , are independent of  $x_1, x_2$ . We compute in turn the TrT and TsT transformations and compare the results at the end.

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<sup>16</sup>This is true if the  $\Gamma^i$  are Hermitian, or anti-Hermitian, which can always be arranged in the cases of interest.

For the TrT transformation we start by performing a T-duality along  $\partial/\partial x_1$ , followed by a *rotation* of the (T-dualized) Killing coordinates  $x_1, x_2$ :

$$(124) \quad \begin{aligned} x_1 &= \cos \theta \tilde{x}_1 - \sin \theta \tilde{x}_2 , \\ x_2 &= \sin \theta \tilde{x}_1 + \cos \theta \tilde{x}_2 . \end{aligned}$$

We then perform a T-duality along  $\partial/\partial \tilde{x}_1$ , denoting the T-dualized coordinates by  $\hat{x}_1, \hat{x}_2$ . The TrT transformed metric and dilaton then read

$$(125) \quad \begin{aligned} ds'^2 &= f^{-1} \left[ f_1 \left[ dx'_1 + adx'_2 + (c - sb)A_1 - s(aB_1 - B_2) \right]^2 \right. \\ &\quad \left. + f_2 \left[ dx'_2 + (c - sb)A_2 - sB_1 \right]^2 \right] + ds_\perp^2 , \\ e^{2\Phi'} &= \frac{e^{2\Phi}}{f} , \quad f \equiv (c - sb)^2 + s^2 f_1 f_2 , \end{aligned}$$

where we have used the shorthand notation  $s = \sin \theta$ ,  $c = \cos \theta$ . The transformed  $B$ -field is

$$(126) \quad \begin{aligned} \hat{B} &= B_\perp - B_1 \wedge A_1 + dx'_2 \wedge [(s + cb)A_1 + c(aB_1 - B_2)] \\ &\quad + f^{-1} [(c - sb)[B_1 - (s + cb)dx'_2] + s f_1 f_2 (cdx'_2 + A_2)] \\ &\quad \wedge [dx'_1 + adx'_2 + (c - sb)A_1 - s(aB_1 - B_2)] . \end{aligned}$$

For the TsT transformation we start again from the solution (123) and perform a T-duality along  $\partial/\partial x_1$ . We then *shift* the (T-dualized) Killing coordinates  $x_1, x_2$ :

$$(127) \quad \begin{aligned} x_1 &= \tilde{x}_1 , \\ x_2 &= \tilde{x}_2 + \gamma \tilde{x}_1 . \end{aligned}$$

Finally, we perform a T-duality along  $\partial/\partial \tilde{x}_1$ , denoting the T-dualized coordinates by  $x'_1, x'_2$ . The TsT transformed metric and dilaton then read

$$(128) \quad \begin{aligned} ds'^2 &= h^{-1} \left[ f_1 \left[ dx'_1 + adx'_2 + (1 - \gamma b)A_1 - \gamma(aB_1 - B_2) \right]^2 \right. \\ &\quad \left. + f_2 \left[ dx'_2 + (1 - \gamma b)A_2 - \gamma B_1 \right]^2 \right] + ds_\perp^2 , \\ e^{2\Phi'} &= \frac{e^{2\Phi}}{h} , \quad h \equiv (1 - \gamma b)^2 + \gamma^2 f_1 f_2 . \end{aligned}$$

The transformed  $B$ -field is

$$(129) \quad \begin{aligned} \hat{B} &= B_\perp - B_1 \wedge A_1 + dx'_2 \wedge [bA_1 + aB_1 - B_2] \\ &\quad + h^{-1} [(1 - \gamma b)[B_1 - bdx'_2] + \gamma f_1 f_2 (dx'_2 + A_2)] \\ &\quad \wedge [dx'_1 + adx'_2 + (1 - \gamma b)A_1 - \gamma(aB_1 - B_2)] . \end{aligned}$$

Let us now compare the results of the two transformations. We see that the metrics agree precisely if we rescale the coordinates of the TrT solution

as  $x'_i|_{\text{TrT}} = \cos \theta x'_i|_{\text{TsT}}$  and identify  $\gamma = \tan \theta$ . The dilatons are then related via

$$(130) \quad e^{2\Phi'}|_{\text{TsT}} = \cos^2 \theta e^{2\Phi'}|_{\text{TrT}} .$$

For the  $B$ -fields one immediately sees that some terms in the two expressions clearly match, but some others apparently differ. A calculation shows that the two expressions are indeed different, but related by

$$(131) \quad B'_{\text{TsT}} = B'_{\text{TrT}} - \sin \theta \cos \theta dx'_1 \wedge dx'_2 .$$

The difference is a closed two-form, and hence the two configurations differ by only a flat  $B$ -field.

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